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"To hold, as 'twere, the mirror up to nature; to show the very age and body of the time."

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Among the challenges of the day, none is being sounded more insistently than the call for more, and more effective, teaching of mathematics.

THE DAYS are not long past since mathematical instruction, beyond a very elementary stage, was decried by many an ostensible educational leader as having little value except for a small minority of the school population. To that judgment we opposed ourselves, stubbornly, but not always with success. We were heard with forbearance, as one hears the defenders of a vested interest, but the place of mathematics in the general curriculum was nevertheless minimized. Other subjects, less intellectually burdened but allegedly more immediately relevant to daily living, were put forward as more appropriate and important.

Time has hastened to disconcert those who carried this issue against us. Today others, without any stimulus on our part, are calling for more widespread mathematical competency, and are doing so with an insistency that astonishes even us. They are telling the public—no, they are warning it—that a great increase in the supply of persons with mathematical training is nothing less than critical for our national safety, for the very preservation of our mode of life. Is this an overcorrection? Is the pendulum which was formerly pushed too far to one side now swinging too far in its return? I thought it might suit this occasion not badly for us to consider this matter, or, in the words of Shakespeare: "To hold, as 'twere, the mirror up to nature; to show the very age and body of the time."

Many a river, as it flows along its course from source to sea, passes in turn through reaches of calm and others of turbulency. Here it glides with a deep and easy sweep, only to plunge into confusion, to tumble and rush chaotically without direction but with much noise toward no apparent goal. The unfolding of history in the stream of time seems sometimes to be of a similar pattern. Quiescent eras, placid and peaceful, apparently sometimes give way precipitately and unaccountably to troublous ones, in which the orders of the day are tensions and conflicts. We find ourselves now in such a disturbed era. Diplomatic and economic crises succeed each other in dismaying tempo, and military ones constantly impend. Our destiny is clouded. Anxiety overshadows us, as we contemplate terrors of war and threats to freedom. Our continued liberty to choose, without hated outside dictation, the cultural patterns and practices to which we shall subscribe, seems suddenly to be in jeopardy. Certainly the historical narrative, as it will be told in the future, will not pass our times by as eventless and dull.

Some of us can remember a calmer past. But when we do so we recall also that the mathematics teacher was not then exalted as he or she now is. Mathematics, at least as a curricular subject, was discounted. It was conceded to have worth only for a few. In the curriculum it was being progressively dislodged from the center, even threatened with complete eclipse. The

yardstick of much educational doctrine was at that time immediate practical utility. Subjects free from baffling difficulties and from the importunities of logic were preferred—reportedly, to protect the pupils' egos from harmful experiences of frustration. The "hard" subjects, mathematics prominently among them, were diluted, softened up, minimized, or displaced.

That such was deluded doctrine, time has hastened to show. The realities of life are not to be conjured away by wishful thinking, nor does the race go to the soft and untrained runner, however much unwarranted self-confidence may have been instilled in him. Now the land rings with calls for sturdy talent, for men and women trained, to whatever stage, but thoroughly, in "hard" mathematics. You, as mathematics teachers, have become objects of national concern, and have been projected into a completely central educational role. What are the reasons for this transfiguration?

The most distinctive traits of our culture—the culture of the West—at least as it has evolved in its later course, have been a devotion to science and the exploitation of science in technology. The origins of these activities lie far in the past. Their dominance as social forces, however, dates back only into the last century. In the more recent past, they have burst forth to sheerly explosive proportions, so that presently they effectively shape and direct our lives. Now mathematics is the mainstay of both science and technology. For it is the medium by which the human genius records its observations upon the structure and behavior of nature, and from which it draws the power to discern the underlying order. It supplies man, in short, with the basis for broadening his understanding and for bending his knowledge to useful ends.

This whole matter of the character of mathematics and its role as a human enterprise is, unfortunately, all too little understood, despite the fact that in a literate society such as ours everyone is at some time exposed to some mathematical instruction. The great majority of people, even of those who are otherwise well informed, are utterly unaware of both the vastness and the diversity of mathematical thought, and of its living, dynamic character. In no measure whatever do they realize that mathematics is the central directive agent of the whole human quest for quantitative understanding of the environment, that it is in command at all points in man's efforts to apply nature to his advantage, that it is the medium through which chaos and superstition are banished, and that it is the backbone not of one science or of several, but of all science.

A scientific doctrine comes to stature and is accorded authority only where, and only to the extent that, it is formulated in mathematical theory. The invention of applicable theories, and their enlargement and perfection, are the mathematicians' demesne. Only where appropriate theories are at hand are the roads to scientific advance and technological development open. The assignment to keep the roads open acts as a continual spur upon mathematics to extend its capability through growth and diversification. Scientific evolution would soon cease in the absence of mathematical expansion. The great technological achievements of our century—transportation by flight through the atmosphere or under the surface of the sea, the mass communication mediums of radio and television, the recording of music, atomic fission and fusion, radar detection and rocket power, the analysis of spatial radiations, the expanded knowledge of the astronomical universe and of the constitution of the earth—all these have, in their appropriate measures, been mathematically achieved.

Fortunately, the inner growth of mathematics, which is thus positively essential, is actually going on. It could even be called vigorous, although, to be sure, it is not often popularly proclaimed like many

more easily apprehended scientific or technological triumphs. New mathematical theories continue to be born and, as lusty infants, to receive widespread attention, while older theories, even venerable ones, are restlessly being rejuvenated and revivified on the basis of new ideas. A single instance of this, from the very immediate past and even from the present, is the development of the high-speed electronic computing machines. These machines are nothing other than mathematical logical structures that engineers have incorporated into hardware. With them mathematics has magnified its own powers so immeasurably in many fields that it can now deal with whole categories of problems that were sheerly beyond its capability—or any human powers—only a score of years ago.

The belief has been voiced that this era in which we now find ourselves will be known in the future as the *technological revolution*, and that its effects will be no less far-reaching and profoundly reforming than were those of the industrial revolution of the eighteenth and nineteenth centuries. It is certainly true that industry is in a revolutionary state. Its production rates and the variety of its commodities, especially of such as were only recently discovered or invented, are unprecedentedly immense, whereas technology has so heightened production efficiencies that increasing tasks are accomplished with ever fewer hours of labor. Routines are being transferred more and more from human hands to machines. In consequence, employment is being shifted in large measure from persons with less to those with more technical training, by which is often meant to those with more knowledge and skill in mathematics.

These changes, even were they confined to our own country, would be deeply affecting. They are not, however, so confined, and by that very fact their impacts are ominously magnified. They have destroyed the bases of former international alignments. New destinies have been born of them, and make themselves felt in pressures and clashes of interests that are burdened with the dangers of war. In the new order which has thus come about, the rise of Russia to the position of a world power is conspicuous. It is a direct result of industrialization. To recognize that fact, is inevitably to recognize also that power among nations will be held by the technologically strongest. In the past the dominant power has been ours. To wrest that from us the Soviets have thrown themselves into the struggle for supremacy in the technological field with a singleness of purpose and an intensity of effort which considerably surpass our own.

The Soviets give their allegiance to political tenets and social ideologies that are poles apart from ours. Their opposition to us is therefore disquieting, and loaded with threats to many of our material interests. And not only to those, but more profoundly to the continuance of our freedom to fashion and direct our own social, economical, political, and religious institutions in the ways we prefer, without alien dictation, and to wield and enjoy justice as we conceive it. With what seems like desperate suddenness we find ourselves in jeopardy of all the devastation that man can loose by his mastery of natural phenomena. For technology, when turned to destruction, operates with terrible effect. We are pitted against an adversary who is swelled with arrogance in a newborn awareness of power. Our doctrines are hateful to him, or at least to his coterie of leaders, and he sees in us the agency which alone threatens the frustration of his gratuitous ambitions. The engagement in which we are thus locked is an unsparing one, in which anything less than our best total effort is certain to be disastrously insufficient.

The Soviets have first-rate scientists and mathematicians. They are resolutely working to increase greatly their supply of trained personnel. The leadership which was ours is slipping away; the prospect of losing it wholly confronts us. In point of

natural resources we are matched; in sheer numbers of men we are completely outmatched. That being so, our hope to maintain an ascendant position must be pinned to the shrewdest possible use of our wits, that only means by which one can magnify oneself to compete with a physical superior. This is what the mirror shows when we hold it up to nature. This is the age and body of the time.

The fact to which we have thus been rudely awakened is that technological progress, that beneficent genie that has heretofore given us our high standard of living, cannot henceforth be safely left to its own recourse, but has become one whose favor we must sue to the utmost and whose welfare it is imperative for us to hasten. Upon that our future depends, and that, therefore, is a matter that impinges quite directly upon all of us. For the future of a country is shaped, in the very first instance, by its educational effort, which is to say by its schools. The burden of a renewed and heightened effort therefore falls upon the schools. And because their strategic importance is clear, we suddenly find the schools admonished of the ominously developing national need in the pronouncements of committees, commissions, and councils, and by advisors to the highest authority of the land. Public attention is being directed upon the schools. Teachers and pupils have become subjects of discussion in circles which in other times rarely gave them a thought. Reappraisals of the schools' responsibilities to the social body are being called for, and the demand is heard for a reorientation of the schools to tougher and more realistic objectives. The need for more teachers is stressed, and the inequities with which teachers have had to content themselves in the past are being acknowledged and promised redress.

Among the challenges of the day none is being sounded more insistently than the call for more, and more effective, teaching of mathematics. From that, every mathematics teacher surely can draw a measure of gratification, for it manifestly emphasizes the importance of the mathematics teacher's job. Personal and professional satisfaction root upon that, but so also does larger responsibility. In countries that are under the sway of dictatorial rule, pupil enrollments in mathematics classes, or, for that matter, in any classes, can be raised by edict and maintained by regimentation. Those expedients are not ours; we neither can nor wish to use them, since we have no remote desire to abrogate each individual's right to choose in matters of his own fortune. Persuasion and guidance are less prompt than compulsion, and are more demanding of patience and devotion. Nevertheless they seem to us to be the preferable means, and to be more certain of honest results.

With pupils exposed to repeated emphasis upon the challenging roles and dynamic character of science and technology through almost uninterrupted broadcasts of news and commentary upon current events and trends, the task of persuading them of the importance of mathematics for many an exciting career should not prove difficult. The main problem may be to correct the belief that such careers are only for supermen. The facts are that opportunities for both men and women with mathematical training, be that through the advanced or the intermediate stage, or even only through the elementary stage, are now numerous, varied, and well rewarded. Careers are open in industry and in many branches of the government, and, of course, also in the schools themselves. Where positions calling for mathematical training were few in the not very distant past, there are now many, and it is foreseen that even a greatly increased supply of trained persons will continue to fall short of the growing demand. In our whole economy the trend is from less trained to better trained personnel.

The call for the more effective teaching of mathematics is being met by efforts, many of them presently under way, to reform the curriculum on the one hand, and

to bolster teacher competency on the other. As to the curriculum, the attention is being directed upon a revaluation of subject matter, with an eye toward minimizing the more dispensable items to permit the introduction of ones that seem to be more relevant to modern needs. The importance of this cannot be questioned. I shall refrain from discussing it, however, both for the reason that I have not personally had a hand in it and because I know that many of your deliberations at this meeting center upon it.

The awakened recognition of the importance of mathematics has, not unnaturally, called forth a demand for higher competencies on the part of its teachers. As a social force, mathematics has changed the times. But it has also changed with the times, and is indeed continually changing. Even in the very recent past, fruitful new ideas in considerable number have been conceived by it, and, on the basis of these, new disciplines have been born and older ones reconstructed. While much of this is recondite and specialized, some of it is, on the contrary, easily accessible and well adapted to enrich teaching backgrounds. The appreciation of this has led to the proffering of teacher institutes, many of them during the summer months, by a number of educationally-oriented agencies. These institutes afford teachers the means both to refresh and extend the stocks of their professional assets. Profit from them will depend, of course, upon the teachers' response. Professional competency is not something that can be embalmed to maintain itself. If it is not continually refreshed, it decays. You, who have had the enterprise to attend this national meeting, are not those who need exhortation upon this point. You are, however, only a small minority of all mathematics teachers, whereas the serious demands of the times will impinge upon all alike.

With the advance of industrialization, the general welfare of society has come to depend more and more upon human acuity and wit. Leadership of high intellectual calibre has become more and more essential in all enterprise, whereas labor and brawn have become more dispensable. Finding ourselves locked, as we now presently do, in an international engagement which foreseeably will strain our every capability to the utmost, and in which failure has a most malevolent aspect, we are faced with the fact that leadership talent is the critical asset above all. The detection, the development, and the proper employment of such talent with the greatest possible promptness and efficiency now confronts us as an imperative.

The call is accordingly going out—again to the schools—to be alert to the early singling out of pupils of superior abilities, and to open ways for their optimal development. The circumstances which are spurring this prod upon the schools are regrettable, but the prod itself I am far from regarding as such. For it restores to a better perspective a matter of which, it has long seemed to me, the schools have not been duly mindful. Equality of educational opportunity for all is a tenet we have enshrined; one to which we will continue to cling. It is mistaken, however, to read into that tenet an implication that pupils must all be treated alike regardless of their unequal capacities to profit therefrom. The total endeavor of the schools is limited by the material and human resources available to them. The maximum return they can give in their role as an organ for the total social good may, therefore, be expected to depend upon a strategic allocation of effort. To consider the keeping-up of the rear guard as the matter of first importance, as a matter to be pressed even at the expense of appropriate attention to the whole column's advance, seems hardly an admirable strategy. Yet it is one to which allegiance has all too widely been given. The schools are being urged now, with considerable insistence, to look more to the head of the column, and when the occasion is ripe, to send out advance guards ahead.

This is not the first time that I have

fallen upon this theme before this National Council. At your summer meeting at Denver, Colorado—I believe it was just ten years ago—I had the honor, as I have it now, to be your banquet speaker. The burden of my message then bore especially upon this matter of the superior student. Some observations I made then are still germane.

I will repeat them here in the identical words. "It is a fact of nature that in any society the genius for leadership resides in only a small minority of the people. But the whole impetus toward advance, the whole implementation of progress, comes from this few. In statesmanship, in science, in education, in the arts, and in business the leaders are the exceptions. The great majority of any population plays no role in shaping its own destiny, but depends inertly upon the leaders, who are the brain of the social body. The best that education can do for the many average pupils would come to little were they not to receive guidance later from the superior minds.

"In recent years we have been drawn more than ever before into international relationships. Our rivals are intelligent, energetic peoples, who are resourceful and determined not to be safely underestimated. They do not see eye to eye with us. That being so, the most elementary considerations must tell us that our best efforts will be needed to hold our own.

"We are pitted against people who have a lively reverence for leadership. In their educational effort they are giving this their foremost attention. Is it the part of wisdom for us to continue our present neglect of our better students?"

The ten years since I wrote those words have subtracted nothing from their truth, but have added greatly to their urgency. The national leadership, which was then ours, has in large measure slipped from our grasp, and has fallen into almost precarious balance. The say on how the beam shall tilt in the future will rest largely with the schools.

The profession of teacher is an exalted one. If life under our institutions and by our standards has values—as I am sure we do believe—those values are summed up in our understanding of the universe, our reverence for its orderly scheme, our faiths, our tenets of conduct and justice, our dedication to freedom for initiative and expressive creation, and our respect for the dignity of the human individual. The orderly transmission of this wealth from the passing generation to the oncoming one, and the inculcation of such respect and love for it as will insure its defense and preservation, that is the teacher's mission. If we hold the mirror up to nature, it shows us that the age and body of the time is one in which our prospects are darkened and our heritage threatened. Teachers—mathematics teachers—if importance of the social niche in which your role is cast inspires you, and who among us is not thereby inspired, yours is a job that is worthy of magnificent effort.

"Substantial changes in the mathematical curriculum are thus long overdue. But a mere change of subject matter is not sufficient. A poor curriculum well taught is better than a good curriculum badly taught. A good curriculum well taught is the only acceptable goal."—"Program for college preparatory mathematics," *Report of the Commission on Mathematics, College Entrance Examination Board, New York, 1959, p. 6.*

The unique factorization theorem[1]

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The unique factorization theorem seems obvious to the beginner, but it is really a deep theorem. Examples presented here should help the teacher show the importance of the unique factorization theorem.

ONE OF THE MOST PERPLEXING aspects of the teaching of mathematics is the presentation of properties which are at once mathematically deep and, to the beginner, "obvious." Probably the most often used property of this type is what is known as the Unique Factorization Theorem, for it is the theorem which gives a basic structure to our number systems of simple arithmetic: the counting numbers and the integers. Moreover, without this theorem there would be no reduction to unique "lowest terms" for fractions—and it follows that the theorem is, then, a cornerstone in the construction of the system of rational, and hence of the real, numbers.

We propose here a means by which this theorem might be appreciated as the deep statement it is, and it may be hoped that material can be selected from this article for adaptation to the classroom at various levels.

We shall discuss the statement of this theorem for the counting numbers and for the integers, then we shall give some examples of systems which do not possess unique factorization. Finally, we shall present a certain class of number systems and discuss unique factorization in these systems.

[1] This article is adapted from a more comprehensive research paper to be published elsewhere. The authors acknowledge the support of the Haverford College Faculty Research Fund.

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STATEMENT OF THE THEOREM

Let us recall, then, a precise statement of the Unique Factorization Theorem. The central concept of the theorem is the concept of a prime; but this concept depends heavily upon the system in which one is working.

Consider first the system of counting numbers: 1, 2, 3, 4, 5, $\cdots$. If $x=y\cdot z$, we say that y and z are factors of x, and in this system, the number 1 is a factor of every number, and every number is a factor of itself. A prime in this system is a number having no factors other than 1 and itself, and we choose to exclude 1 from the list of primes (for reasons which shall be explained shortly). Hence, we may define a prime in the system of counting numbers as a counting number with precisely two unequal factors, and the first few primes are 2, 3, 5, 7, 11, 13, 17, 19. An explicit statement of the Unique Factorization Theorem for counting numbers is:

Every counting number other than 1 *is either itself a prime or it can be factored uniquely as a product of primes.*

(The order in which the factors occur may of course vary, but since multiplication of counting numbers is commutative, this is irrelevant.)

In practice, this theorem states that no matter how one proceeds in breaking down a counting number with respect to multiplication, the ultimate factorization al-

ways possesses the same prime factors. For example, one might factor 420 as

$$420 = 14 \cdot 30$$

or as

$$420 = 20 \cdot 21.$$

Upon continuing this process as far as possible, we obtain

$$\begin{aligned} 420 &= 14 \cdot 30 \\ &= 2 \cdot 7 \cdot 3 \cdot 10 \\ &= 2 \cdot 7 \cdot 3 \cdot 2 \cdot 5 \\ &= 2 \cdot 2 \cdot 3 \cdot 5 \cdot 7 \end{aligned}$$

and

$$\begin{aligned} 420 &= 20 \cdot 21 \\ &= 2 \cdot 10 \cdot 3 \cdot 7 \\ &= 2 \cdot 2 \cdot 5 \cdot 3 \cdot 7 \\ &= 2 \cdot 2 \cdot 3 \cdot 5 \cdot 7 \end{aligned}$$

It is now easy to see why we exclude 1 from the list of primes, for otherwise $420 = 1 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 7$ would be another factorization of 420 into primes, while $x = 1 \cdot x$ is always considered a trivial factorization.

Various proofs of this theorem are readily available in standard texts,[3] and, in any case, our concern here is with the content —not the proof—of the theorem.

The statement of this theorem is a bit more involved for the system of integers: $\cdots -4, -3, -2, -1, 0, 1, 2, 3, \cdots$. Indeed, the concept of a prime is slightly more complicated for the integers than for the counting numbers. In the first place, every integer is a factor of 0 (since $0 = 0 \cdot x$ for every integer x). Further, every integer x has $1, -1, x, -x$ as factors. Excluding 0, 1, and -1 from the list of primes, we define a prime integer as an integer with precisely four unequal factors. Some primes are ± 2, ± 3, ± 5, ± 7, ± 11.

Now it should be obvious that we must, for the integers, modify the statement we made previously of the Unique Factorization Theorem, for note that $6 = 2 \cdot 3$ and $6 = (-2)(-3)$. These are two distinct factorizations of 6 as a product of primes—but the primes differ only by sign, and we allow ourselves to hedge on just this point! The Unique Factorization Theorem for integers states:

> *Every integer except* 0, 1, *and* -1 *is either itself a prime or it can be factored as a product or primes, and this factorization is unique except for the signs of the factors.*

(Again, we do not worry about the order of the factors.)

The Unique Factorization Theorem for the integers is an immediate consequence of the Unique Factorization Theorem for the counting numbers—so immediate, in fact, that its proof could be assigned as an exercise for students.

Some easy examples

From the above discussion, two things need to be noted—one about the concept of a prime and one about the statement of the Unique Factorization Theorem. Take note that primes are simply those numbers which cannot be further broken down in a nontrivial way in the system with respect to multiplication, and we shall hold fast to this concept. Also take note that the statement of the Unique Factorization Theorem involves only one algebraic operation—multiplication. Thus, in our examples we shall insist on multiplicative closure, i.e., we shall insist that the product of any two numbers in a system again be in that system.

Now consider the following six sets of numbers:

OC: $1, 3, 5, 7, \cdots, 2x+1, \cdots$

EC: $2, 4, 6, 8, \cdots, 2x, \cdots$

RC: $1, 4, 7, 10, \cdots, 3x+1, \cdots$

OI: $\cdots, -5, -3, -1, 1, 3, 5, 7, \cdots, 2x+1, \cdots$

EI: $\cdots, -6, -4, -2, 0, 2, 4, 6, \cdots, 2x, \cdots$

[3] R. A. Beaumont and R. W. Ball, *Introduction to Modern Algebra and Matrix Theory* (New York: Rinehart, 1954), p. 4.

RI: $\cdots, -8, -5, -2, 1, 4, 7, 10, \cdots,$
$3x+1, \cdots$

The set OC is simply the set of odd counting numbers, i.e., the set of counting numbers excluding multiples of 2. The product of any two odd counting numbers is odd, and the primes in this system are 3, 5, 7, 11, 13, $\cdots$; i.e., the primes are just the same as in the counting numbers except for the prime 2. This being the case, if it were possible to express a number in this system in more than one way as a product of primes, then of course the same would be true in the system of counting numbers; consequently, the system OC does enjoy unique factorization.

The set EC is the set of even counting numbers—each is a multiple of 2. The product of any two even counting numbers is an even counting number, and since no counting number of the form $4x-2$ can be written as the product of even counting numbers, the primes of this system are 2, 6, 10, 14, 18, 22, $\cdots$. That this system does not possess the property of unique factorization can be seen by considering 36:

$$36=2\cdot 18$$

and

$$36=6\cdot 6,$$

two distinct factorizations into primes.

The set RC comprises an example which was brought to our attention by the famous mathematician Hans Rademacher.[4] The product of any two elements of the set RC is again in RC, for if $3x+1$ and $3y+1$ are two counting numbers of this type, then their product is $3(3xy+x+y)+1$. The primes of this system are 4, 7, 10, 13, 19, 22, 25, 31, $\cdots$; and a little thought on the matter tells us that the primes of RC are either counting number primes of the form $3x+1$ (e.g., 7, 13, 19, 31, $\cdots$) or are the product of exactly two counting number primes of the form $3x+2$ (e.g., 4, 10, 22, 25, $\cdots$). That this system does not have unique factorization is shown by the fact that 100 factors in two different ways as the product of primes in the system:

$$\begin{aligned}100&=4\cdot 25\\&=10\cdot 10.\end{aligned}$$

The set OI, the odd integers, does have unique factorization with the proviso that we ignore—as in the integers—the signs of the factors. The set EI, the even integers, does not have unique factorization. These two statements we leave for verification by the reader.

However, in contrast to RC, consider RI, the set of all integers of the form $3x+1$. The product of two such integers is again such an integer. Note that $4=(-2)(-2)$ and is hence no longer a prime, as it was in the set RC. Indeed, if p is a positive prime integer of the form $3x+2$, then $-p$ is in the set RI. Thus, we do have unique factorization here, and strongly so. That is to say, we are not forced to state the theorem for this system with the usual stipulation that signs of the factors do not matter. Everything other than 1 and primes factors uniquely as a product of primes, the signs of the factors also being uniquely determined. For example,

$$100=(-2)(-2)(-5)(-5),$$

and no other factorization into primes of the system exists.

General results

We have thus far attempted to show by examples (especially by the sets RC and RI) that the Unique Factorization Theorem is a deep theorem, requiring careful consideration. (To see its far-reaching effects, try to construct "rationals" by taking quotients of elements from the system RC. Note that 40/100 can be expressed as

[4] Examples of this type exist in various places in the literature and were well known by the early investigators of algebraic number theory. See, for example, Legh Wilber Reid, *The Elements of the Theory of Algebraic Numbers* (New York; Macmillan, 1910), p. 254.

a reduced fraction in two different ways:

$$\frac{40}{100}=\frac{4\cdot 10}{10\cdot 10}=\frac{4}{10}$$

and

$$\frac{40}{100}=\frac{4\cdot 10}{4\cdot 25}=\frac{10}{25}\ .$$

Also, 4 and 10 are both factors of 100, and 4 and 10 are relatively prime in this system, but their product, 40, does not divide 100. We could proceed to exhibit other unusual properties, but to do so is not the purpose of this article.)

Now let us consider some general systems. Consider, first, all counting numbers of the form $Ax+B$, where A and B are fixed integers, $A>0$, $A>B\geqq 0$. The authors have recently shown the following to be true:

> *The set of all counting numbers of the form* Ax+B *is closed with respect to multiplication and possesses unique factorization if and only if* B=1 *and* A=1 *or* 2.

In other words, the counting numbers and the odd counting numbers are the only such sets with unique factorization. The proof is complicated and will appear elsewhere as part of a more inclusive work on number theory by the authors. We shall, however, give a construction for a number which does not factor uniquely in the system of counting numbers of the form $Ax+1$ when $A>2$. Simply note that

$$\{A(2A-3)+1\}^2=\{A(A-2)+1\}\cdot\{A(4A-4)+1\},$$

and except for the two cases $A=5$ and $A=8$, each of the factors involved in this factorization is a prime of the corresponding system. For $A=5$ and $A=8$, unique factorization is denied by

$$21\cdot 26=6\cdot 91$$

and

$$33\cdot 33=9\cdot 121,$$

respectively.

Next, consider all integers of the form $Ax+B$, with A and B fixed, $A>0$, $A>B\geqq 0$. For these systems, we have shown:

> *The set of all integers of the form* Ax+B *is closed with respect to multiplication and possesses unique factorization if and only if* B=1 *and* A=1, 2, 3, 4, *or* 6.

This tells us that the integers, the odd integers, the set RI, and two others are the only arithmetic sequences of integers possessing unique factorization. Again, the proof will not be presented here. The case $A=1$, $B=1$ gives just the integers, and they have unique factorization; the case $A=2$, $B=1$ gives the odd integers OI; the case $A=3$, $B=1$ gives RI, and we have discussed this earlier. For $B=1$ and $A=4$ or 6, the reader will have little difficulty in establishing the unique factorization property. For the other cases, one can —by experimentation—readily gain experience to find numbers which do not factor uniquely. It is instructive to play with such examples as classroom exercises. We give three such:

(i) When
$A=5$, $B=1$, $36=6\cdot 6=(-4)(-9)$

(ii) When
$A=7$, $B=1$, $120=8\cdot 15=(-6)(-20)$

(iii) When
$A=8$, $B=1$, $225=9\cdot 25=(-15)(-15)$.

As a final remark, note that—as in the set RI—the cases $A=4$, $B=1$, and $A=6$, $B=1$ lead to a strong statement of the unique factorization property wherein we are not obliged to hedge about the signs of the factors.

"In questions of science the authority of a thousand is not worth the humble reasoning of a single individual."—*Galileo.*

Clock arithmetic and nuclear energy

FRANCIS SCHEID, *Boston University, Boston, Massachusetts.*

A particle enters a sheet of lead. It collides with an atom of lead and bounces off in a new direction toward another collision. What per cent of the number of particles entering the lead will succeed in penetrating it? Clock arithmetic helps find the answer.

HOW MUCH IS $6+7$? Why, 13, of course. But wait a minute, that's not so on a clock! Seven hours after six o'clock it's *one* o'clock. So you could say that on a clock $6+7=1$. The clock just throws away the other twelve. In the same way you would have $7+7=2$ and $11+9=8$. This simple idea of throwing away the twelve leads to what mathematicians call addition *modulo 12*. Notice that in addition modulo 12

$$1+12=1$$
$$2+12=2$$
$$3+12=3.$$

Whatever is added to 12 comes out unchanged. In ordinary arithmetic whatever is added to *zero* comes out unchanged. So it might not be a bad idea to say 0 o'clock instead of 12 o'clock. Clock arithmetic would then use only the numbers 0, 1, 2, · · · , 11.

There are arithmetics which use even fewer numbers. For example, to add modulo 4 you simply throw away 4 whenever necessary to make the result either 0, 1, 2, or 3. The appropriate addition table is shown in Table 1.

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2.

TABLE 1

To see how the table works follow the two arrows and find that $2+2=0$, which is *correct modulo 4*. You may want to check one or two other values yourself, or to construct a similar table for adding modulo 12.

Now, a reasonable question is, "If we can add this way, why not *multiply*?" And indeed, why not? Here is the table for multiplying modulo 4.

	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

TABLE 2

Let us check the two circled results. You know that multiplication is really just addition disguised a little bit. For example, 3×2 means $2+2+2$ (or, take 2 three *times*). So to do our multiplication we must go back to the table for *adding* modulo 4. Let us see what we get.

$$3\times2=2+2+2=(2+2)+2=0+2=2$$

(Don't forget that $2+2=0$.) So $3\times2=2$ as the table says.

Now try 3×3. It goes like this:

$$3\times3=3+3+3=(3+3)+3=2+3=1.$$

Again we agree with the table.

By this time you must be wondering what all this has to do with nuclear energy. Well, be patient just a little longer. We have one more thing to do first, and

TABLE 3

NUMBER OF TRIALS	NUMBER OF SUCCESSES	PERCENTAGE OF SUCCESSES
10	5	50.0
20	7	35.0
30	11	36.7
40	14	35.0
50	16	32.0
60	17	28.3
70	22	31.4
80	26	32.5
90	28	31.1
100	31	31.0

it will not seem to have any connection at all with what we have just done.

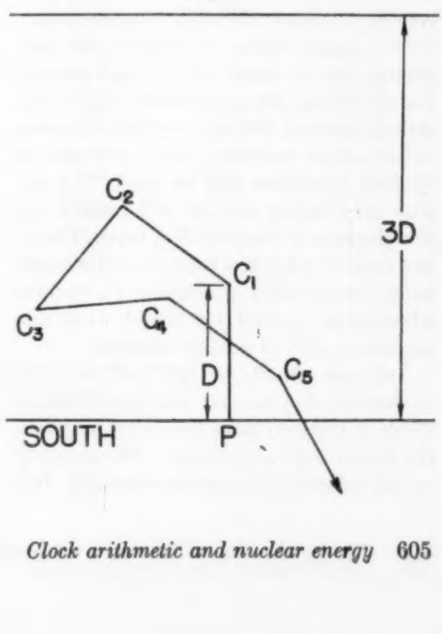
Suppose that a Western Union messenger is trying to make a fast dash across Times Square on New Year's Eve. He starts out at high speed from one side and soon collides with a happy reveler. He bounces away (at reduced speed) and soon crashes into another. Again he bounces off, and so on. After a number of collisions he has lost most of his speed and finds himself drifting slowly around with the crowds. Now, the question is "What were his chances of getting through?"

An answer could be found experimentally by actually making some trials (next December 31) and observing what fraction is successful. Such an experiment would, however, spoil a lot of good fun and might break a few bones. We can perform a similar experiment with pencil and paper and a simple free-spinning arrow made of cardboard. To avoid complications let us assume that the distance moved between collisions is always the same, call it D. Starting at point P, perhaps on the south side of the square, we send our "messenger" (see the path in Figure 1) directly into the crowd, a distance D, to point C_1 where the first collision occurs. Since we do not know in what direction he should rebound we now choose one *at random* by pivoting our arrow at C_1 and giving it a good spin. When it stops we choose the direction in which it points for the next move. (This procedure may not be a perfect model of the real one, but it certainly seems like a good approximation. It amounts to assuming that no direction of rebound is more likely than any other.) Our messenger is now at point C_2 and suffers another collision. Pivot the arrow at C_2, and it again selects a random direction of rebound. The messenger follows the arrow to C_3 where he has his third collision, and the experiment continues in this fashion. If at any time our man comes back out of the crowd on the south side a failure is counted, and a new trial is begun at P. On the other hand, coming out on the north side counts as a success. Finally, suppose that ten collisions are all he can stand. Then if C_{11} is still not a success, we score a failure.

Make a few trials yourself. You'll get some surprises. I made 100 while watching TV the other night. The width of my crowd was 3 times D. The first trial is shown in Fig. 1 and was a failure. The results after 10 trials, 20, 30, and so on, are listed in Table 3. About 31 per cent were successful in getting through.

Now, to bring nuclear energy into the picture, suppose the south side is the inside of a nuclear reactor, with thousands of tiny particles flying around at high speed.

Figure 1



01
13
69
97
61
93
09
17
21
73
49
37
81
53
89
57
41
33
29
77
01
13
etc.

TABLE 4

And suppose our Times Square is really a thick layer of lead or some other shielding material. We can imagine a particle as it enters the layer, collides with an atom of lead, bounces off in a new direction, has another collision, and so on. Each collision may slow it down a little, and after a while it stops. Is not this problem very much like our last one (with nuclear particles instead of messengers)? How many tiny particles will get through? In other words, how effective is the shield in preventing outward radiation of nuclear particles? Again, there are reasons for preferring not to make actual experiments. For one thing, the experiments might be a little dangerous. But our method of tracing make-believe particles which rebound in random directions can be used. The per cent penetrating *through* will predict the effectiveness of the shielding layer. This is just exactly what has been done (for much more complicated problems) in various laboratories around the world. It is one important part of nuclear research.

And now, finally, to get back to clock arithmetic. Let us take one more look at Table 3. Column three lists what we want, the percentage of successes. The numbers in this column fluctuate considerably. It is not easy to choose our answer. The best guess may be the last value, 31 per cent. If we needed more accurate answers what would we have to do? Of course—*run more messengers.* Actually, there are ways of calculating how many it would take to get answers to certain accuracy. For two correct digits we would need about a thousand trials. Can you imagine yourself spinning the arrow ten thousand times? I cannot. Then how can so many trials be made? The answer is "on a modern high speed automatic *computing machine*"*!* The whole job will be done in less than a minute. *But,* a machine will not spin arrows. It needs a different method for choosing random directions, a method that is adapted to its own strong point, namely, high-speed arithmetic. It must be taught to choose *random numbers.*

To explain what we mean by random numbers and to illustrate how they can be produced, let's imagine for simplicity a machine that handles only two digit numbers, like 13 and 99. (Most machines deal with numbers of ten digits or more.) Let's start with the number 1 or (01) and multiply by 25. Of course we get 25. Now multiply by 25 again—that's 625. *Throw away all hundreds.* Or in other words *multiply modulo 100.* You now have 25 again. If you keep doing this, the sequence 1, 25, 25, 25, 25, · · · appears. Certainly these are not random numbers. The choice of 25 as multiplier was very poor for this machine. But now try 13. Remember to multiply modulo 100.

$$1\times 13=13$$
$$13\times 13=69$$
$$69\times 13=97$$

and so on. The entire list of products is shown in Table 4. There are 20 different numbers. After 77 the list repeats. Notice that these numbers are well distributed. Five of them are between 00 and 24, five more between 25 and 49, another five between 50 and 74, and the rest between 75 and 99. (Check for yourself how many fall between 00 and 09, or 10 and 19, etc.)

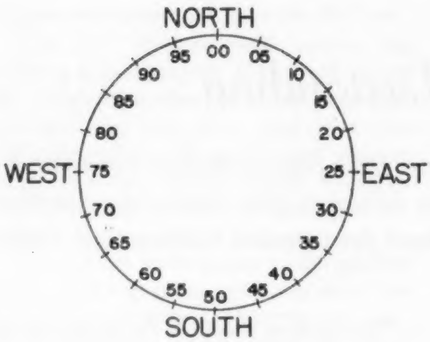


Figure 2

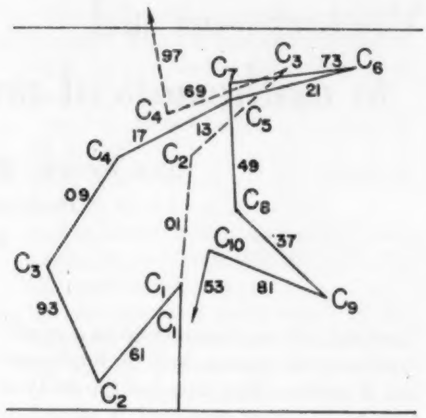


Figure 3

This even distribution is part of the idea of randomness.

Now we could tell our machine to produce these numbers one by one, and to interpret each as a direction. If 00 means North; 25, East; 50, South; and 75, West, as in Figure 2, then our twenty numbers represent twenty "in between" directions scattered somewhat at random around the compass. Starting with 01, for example, the path shown by the broken line in Figure 3 would be traced, using the numbers in our list down to 97. Starting the next trial with 61, the path shown by the solid line in Figure 3 would be followed. One of these "messengers" gets through. Of course, our list is much too short to make very many trials, but a real machine working with, let us say, ten-digit numbers can multiply modulo 10,000,000,000 and have plenty of numbers available. The results of 17,747 runs on an IBM 704 at the Massachusetts Institute of Technology are shown in Table 5, which is certainly a considerable improvement over the earlier one. Notice, however, that the figures to the right of the decimal point are still in doubt. Fluctuations are still large in these positions. To obtain two-decimal-place accuracy we would need about ten million trials. And this would take perhaps 100 hours of time even on the fastest computer. Fortunately, such accuracy is not yet needed, and the method of clock arithmetic and random numbers is producing useful results for our nuclear scientists.

TABLE 5

NUMBER OF TRIALS	NUMBER OF SUCCESSES	PERCENTAGE OF SUCCESSES
1780	506	28.43
3555	1023	28.78
5345	1527	28.57
7142	2027	28.38
8931	2518	28.19
10698	3012	28.15
12443	3500	28.13
14208	4017	28.27
15998	4523	28.27
17747	5044	28.42

"He is unworthy of the name of man who is ignorant of the fact that the diagonal of a square is incommensurable with its side."—*Plato*.

Vectors—an aid to mathematical understanding

DAN SMITH, *Wilton High School, Wilton, Connecticut.*

Instruction in vectors offers many opportunities to teach fundamental mathematical ideas.

VECTORS CAN BE INTRODUCED as a purely mathematical system, built on fundamentals of algebra. This aspect of the study of vectors has been neglected in secondary mathematics.

Vectors are defined in this paper as an ordered set of numbers, and rules of operation are given. This is not the customary introduction to vectors, which is graphical and intuitive. However, it complements this customary approach, and has advantages which make it worthy of consideration.

Some of the advantages found in this mathematical approach to vectors are as follows:

1. It affords practice in adding and subtracting signed numbers; many students do not perform such operations as "$3-(-5)$" correctly, even in their second year of algebra.
2. The use of vectors of more than three components illustrates the abstract nature of mathematics; the same rules of operation are followed although there is no graphic model.
3. Commutativity and associativity can be illustrated through this approach as well as by the graphic treatment. More advanced classes may go on to the fact that vectors constitute a group under addition.
4. The student meets a situation where multiplication is not defined intuitively; he has a chance to consider the nature of multiplication, and more than one type may be demonstrated.
5. The student practices the graphing of ordered pairs; he touches on solid geometry in graphing vectors with three components.
6. The relationship between rectangular and polar co-ordinates can be brought out; complex numbers can be reviewed (or previewed).

Although this approach to vectors limits the student to centered vectors in the beginning, the better understanding of the computational aspects of vectors extends his potential understanding of the subject. On the other hand, such a purely mathematical introduction may be criticized in that it does not motivate the study of vectors, where the intuitive graphic approach does.

The program which follows is designed under the assumption that mathematics can be interesting in itself if the material is presented correctly. In this program the student proceeds at his own pace through a series of fill-in statements. These statements are so designed that only occasional mistakes will be made. When the student finishes, he has grasped the fundamentals of the subject more firmly because he has thought through them in his own way.

This program has been used with success in classes with freshmen, sophomores, and juniors. Since it is an introduction it does not take up multiplication. The students who used this program were not introduced to the subject ahead of time. This program is but one step in a series of revisions, each based on the results obtained from a previous version.

Instructions to the Student for a Self-Teaching Unit on Vectors

This is a "do-it-yourself" method for learning a new subject. If you will follow the simple directions here, we hope you will find that it is easy, and that you will learn quickly. The idea is that you can go at your own speed; nobody will hold you back, and nobody will leave you behind.

You will not write on any of the printed sheets that are given you; you write on your own answer sheet. Thus you can keep the printed sheets and use them again and again for review.

You will be given a sheet of paper to use for your own answer sheet. Place it so that it covers the printed answer sheet. You will uncover the printed answer to a statement only after you first write it down yourself on the graph paper; in this way, you will have to think for yourself, but you will not have to wait to be checked.

Read the first statement, write on your answer sheet the word you think should go in the blank or blanks. Then check your answer by uncovering one answer on the answer sheet; keep the next ones covered.

If your answer is right or wrong, go on to the next statement. However, if it is wrong, first write the number of the question at the top of your answer sheet.

After you finish all statements, start over, doing *only* those which you got wrong. When you have answered all statements correctly *once*, you are through.

The statements are supposed to be easy. You will probably make very few mistakes. Please do your own work, and list the mistakes you make. This will not count on your grade; you do not even have to put your name on the paper.

Please note the time you began and the time you personally finished on the answer sheet, so we can tell how people differ in speed. Please do not study the sheets again until you have been quizzed twice. The quizzes will not count on your grade. We want to be able to judge how well you learned this material with one reading. Thank you in advance for your help.

1. Vectors are used to describe quantities.
2. A vector is an ordered set of numbers. (3, 4, 6) is a vector. (5, −7, 9, 2) is also a ________.
3. (−6, 2) is a ________; so is (4, 5, −8).
4. (3, 4, 6, 2) is a ________; (x, 7, z, w, m, 5, n) is a ________.
5. A ship travels 6 miles east and 2 miles north; its position is (6E, 2N) or (6, 2); this number pair is a ________.
6. An airplane is 6 miles south of its hangar, 2 miles west, and $\frac{1}{2}$ mile up in the air; its position can be given as (5, 2, $\frac{1}{2}$). This number triple is a ________.
7. The position of a point on a graph is given by an ordered pair of numbers such as (4, −6). This pair can be considered to be a ________.
8. The numbers which make up a vector are called its "components"; in the vector (6, −7, 5, 3, 9), the number "−7" is the second ________, and the number "9" is the ________ component.
9. The vector (−3, 5) is made up of two components, −3 and ________.
10. A vector can have more than two components. (−6, 5, 4) has three, (a, b, c, d) has ________.
11. (x, y, z, w, v) is a ________ which has five ________.
12. Figure 1 shows a graphic representation of a vector. For simplicity, we say "it represents the vector," or "it *is* the vector." With this understanding, we can say that this is the vector (3, ____).
13. Figure 2 shows the vector (−4, ____).
14. Figure 3 shows the vector (____, ____).
15. Figure 4 shows the vector (____, ____).

Vectors with three components can be given a graphic representation also; we will not show this in the first lesson. Vectors with more than three components, however, cannot be shown on a graph.

16. Vectors can be added. (a, b, c) plus (x, y, z) equals ($a+x$, $b+y$, $c+$____).

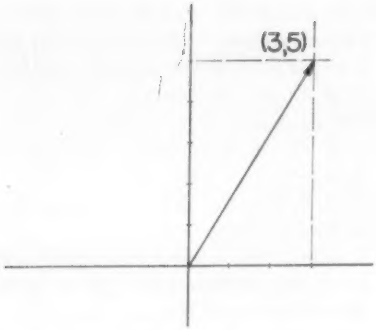


Figure 1

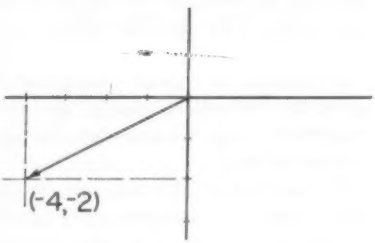


Figure 2

17. $(3, -2)$ plus $(6, 8)$ equals $(9, \_\_\_)$. This sum is another ______.
18. $(5, 6, -4)$ plus $(-3, 2, 7)$ equals $(2, 8, \_\_\_)$; this sum is another ______.
19. (x_1, x_2, x_3, x_4) plus (y_1, y_2, y_3, y_4) equals $(x_1+y_1, x_2+y_2, x_3+y_3, \_\_\_+\_\_\_)$.
20. $(2, 4)$ plus $(5, 2)$ equals $(2+5, 4+2)$ or $(\_\_\_, \_\_\_)$. Note that the sum is a vector, and is represented in Figure 5.
21. $(-2, 3)$ plus $(5, 1)$ equals $(\_\_\_, \_\_\_)$. Note that the sum vector is also shown in Figure 6; it is the diagonal of a parallelogram whose sides represent the other two vectors.
22. Draw a rough graph on your answer sheet; sketch the vectors $(3, 5)$ and $(-4, 2)$; then sketch their sum.
23. Vectors can be subtracted as well as added. $(a, b, c)-(x, y, z)=(a-x, b-y, \_\_\_)$.
24. $(5, 7)-(3, 2)$ equals $(5-3, 7-\_\_\_)$ or $(\_\_\_, \_\_\_)$.
25. $(6, -3, 5, -4)-(4, 2, -3, -5)$ equals $(6-4, -3-2, 5-(-3), -4-(\_\_\_))$;

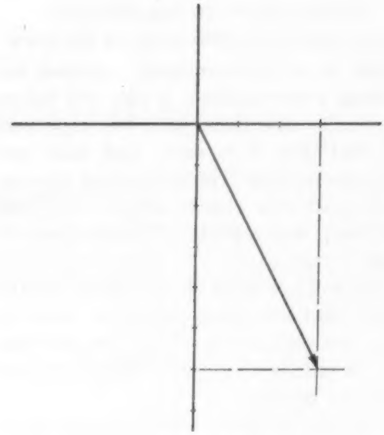


Figure 3

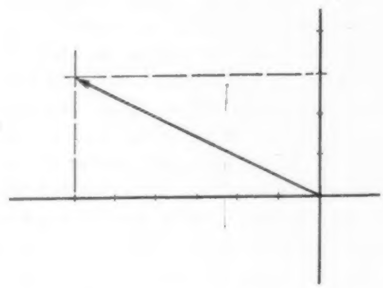
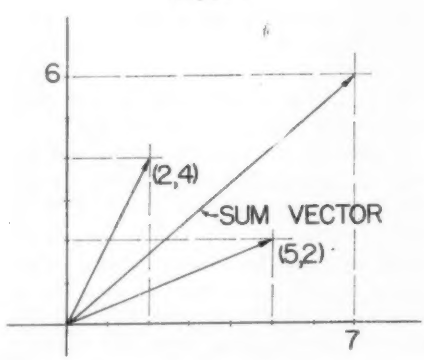


Figure 4

Figure 5



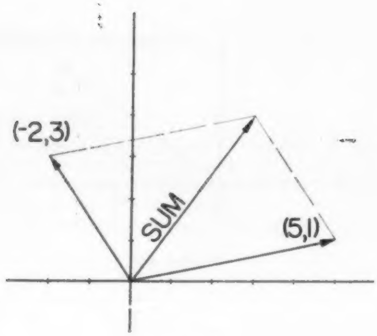


Figure 6

when simplified this equals (2, −5, ____, ____).

26. (−7, 6) − (3, 2) equals (____, ____). Notice that the answer, or difference, is another vector, and is represented in Figure 7.

27. (4, 1) − (2, 3) equals (____, ____). Notice that the graph (Fig. 8) of the difference vector is equal in length to the dotted line drawn between the arrowheads.

28. (−1, −4) − (−3, 2) equals (____, ____). Notice that the graph (Fig. 9) of the difference vector is equal in length to the dotted line between the arrowheads, and that it is parallel to it also.

29. (−3, −4) − (6, −5) equals (____, ____). Notice again that the graph (Fig. 10) of the difference vector could have been found simply by drawing a line between the arrowheads, and then drawing a line parallel to this through the origin. The length of the difference vector is the same as the distance between the arrowheads.

30. Draw a rough graph on your answer sheet, and sketch the vectors (3, −5) and (−5, −3). Then, subtract the second from the first, and sketch the graph of the difference (which is a vector).

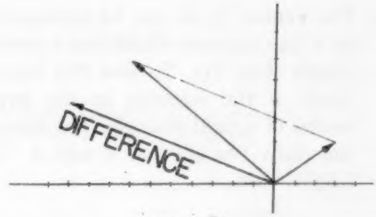


Figure 7

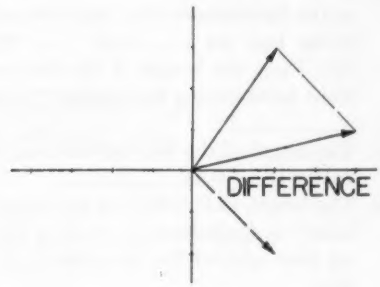


Figure 8

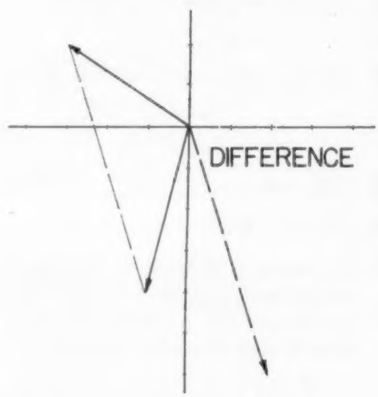
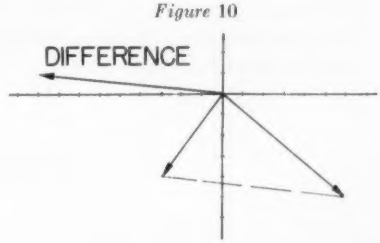


Figure 9

Figure 10



31. The vector (4, 3) can be represented by a line segment which has a certain length (Fig. 11). To find this length, think of the segment as the hypotenuse of a right triangle. The sides of the right triangle are 4 and 3. The length, then, is:

$$\sqrt{4^2+3^2}=\sqrt{\_\_\_\_}=\_\_\_\_.$$

32. The vector $(-2, 5)$ can be represented as the hypotenuse of a right triangle whose legs are ____ and ____ (Fig. 12). Thus, the length of the line segment representing the vector $(-2, 5)$ is ________.
33. The length of the line representing the vector $(6, -4)$ is ________.
34. The length of a vector, or its "magnitude," is symbolized by putting vertical lines around the components, like this:

$$|(3, -4)|=\sqrt{3^2+(-4)^2}=\sqrt{25}=5.$$

35. $|(2, 7)|=\sqrt{2^2+7^2}=\sqrt{4+49}$

$$=\sqrt{\_\_\_\_}.$$

36. $|(-4, -5)|=\sqrt{(-4)^2+(-5)^2}$

$$=\sqrt{\_\_\_\_+\_\_\_\_}$$

$$=\sqrt{\_\_\_\_}.$$

37. $|(3, -6)|=$________.
38. $|(-1, -1)|=$________.
39. A vector is sometimes indicated by using one letter; to show that this letter stands for a vector, an arrow is drawn over the letter like this:

$$\vec{p}=(3, -6) \quad \vec{a}=(4, 3, -2, 7).$$

40. Sometimes, in order to save printing costs, a vector is symbolized by a letter in heavier print than the other letters used; however, we will continue to use the arrow.

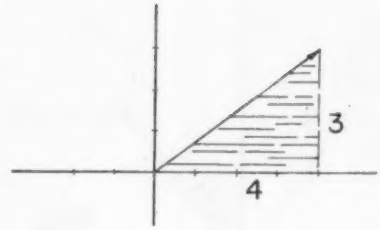


Figure 11

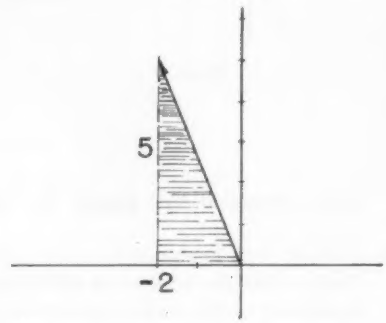


Figure 12

41. $\vec{p}=(3, 4) \quad y=(-1, 5)$

$\vec{p}+\vec{y}=(\_\_\_\_, \_\_\_\_).$

42. $\vec{p}=(7, -2) \quad \vec{x}=(-3, 2)$

$\vec{p}-\vec{x}=(\_\_\_\_, \_\_\_\_).$

43. $\vec{a}=(2, -4) \quad |\vec{a}|=$ ____.
44. This program has introduced you to some mathematical ideas about vectors. Vectors have many practical applications in science, but since this is only an introduction, we cannot go into them here.
45. Please go back and do over the statements you missed. When you have done each one correctly once, you are finished with the program. Then please note the time at which you finished, and write it on your answer sheet. Thanks for your co-operation.

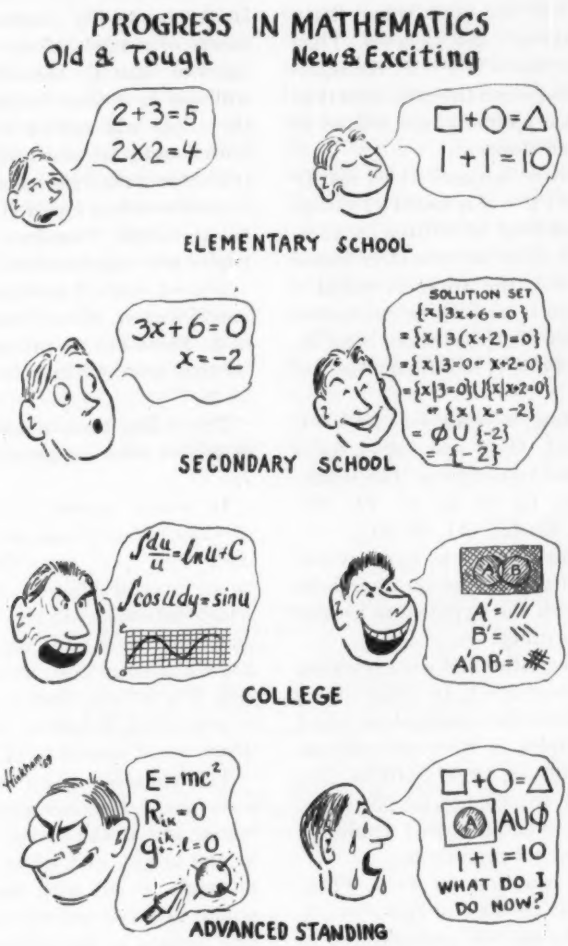
Letter to the editor

Dear Editor:

A couple of Sundays ago I decided to amuse myself by drawing a cartoon as a caricature of the old and new in mathematics. I am enclosing a copy of the resulting cartoon.

You realize, of course, that this is a tongue-in-cheek situation as no one can be this reactionary in today's program.

Very truly yours,
J. J. WICKHAM
Yonkers, New York



Parabolas and Pythagorean triples

E. W. GRUHN, *Orange County Community College, Middletown, New York.*

At their intersection, a cone, a cylindrical surface, and planes have co-ordinates which may be Pythagorean triples.

LET $x^2 = 2ay + a^2$ be the equation of a parabola. Add y^2 to both sides. Then $x^2 + y^2 = y^2 + 2ay + a^2$. Let $z = y + a$. Then $z^2 = y^2 + 2ay + a^2$ and $x^2 + y^2 = z^2$, the equation of the Pythagorean theorem. It is thus seen that certain parabolas are related to the Pythagorean theorem.

A set of positive integers which satisfy the equation $x^2 + y^2 = z^2$ is called a Pythagorean triple and may be written $(x, y; z)$. If x, y, and z have no common factor greater than one, the triple is called a primitive. There is an infinite number of triples. Not only that, but there is an infinite number of sets of infinite numbers of triples.

In the equation, $x^2 = 2ay + a^2$, let $a = 1$. Then, $x^2 = 2y + 1$. Only *odd* values for x will give integral values for y. The resulting triples are: (3, 4; 5), (5, 12; 13), (7, 24; 25), (9, 40; 41), (11, 60; 61), $\cdots$. Note that x must be odd as $2y + 1$ is odd for all values of y. Thus we get an infinite set of triples with the hypotenuse greater than one leg by unity.

In the same equation, let $a = 2$, yielding $x^2 = 4y + 4$ or $y = x^2/4 - 1$. In order that y be integral, x^2 must be a multiple of 4 and $x^2 \geq 16$. The triples in this case will be: (4, 3; 5), (6, 8; 10), (8, 15; 17), (10, 24; 26), (12, 35; 37), (14, 48; 50), (16, 63; 65), $\cdots$. Note that the even-numbered triples in the sequence are not primitives.

In the same equation, let $a = 3$. Then, $x^2 = 6y + 9$ or $y = (x^2 - 9)/6$ or $2y = x^2/3 - 3$. Here x^2 must be an odd multiple of 3 in order that the right member be even since $2y$ is even. The triples are: (9, 12; 15), (15, 36; 39), (21, 72; 75), (27, 120; 123), $\cdots$. It can be readily shown that if two elements of a triple have a common factor (greater than 1), then the third element will also have that factor, and, therefore, the triple will not be a primitive. Here both x and y are multiples of 3, for, since 3 is a factor of the right member of $x^2 = 6y + 9$, it must also be a factor of x^2 as $2y + 3$ must be an integer. Therefore, for $a = 3$, all the triples are nonprimitive.

If $a = 4$, we will have $y = x^2/8 - 2$, and x must be even. Also, x^2 must be a multiple of 8. There are no primitives in this set, for they are the triples for $a = 2$ multiplied by 2.

Proceeding thus, additional infinite sets of triples may be generated for $a = 5$, 6, 7, $\cdots$.

It would appear that the equation $x^2 = 2ay + a^2$ will lead to *all* the triples, if $a = 1, 2, 3, \cdots$, since there must always be an integral difference between z and y, which difference will be a value of a. If a is odd, then a^2 is odd. $2ay$ is always even, $2ay + a^2$ is then odd, and x must then be odd. If a is even, then a^2 is even, $2ay + a^2$ is even, and, therefore, x^2 must be even. Thus, a and x must be of the same parity.

The equation $x^2 + y^2 = z^2$, in three dimensions, determines a right circular cone whose axis is the z-axis, whose vertex is at the origin, and whose elements make an angle of 45° with the z-axis. All the points whose co-ordinates are triples are on this surface in the all-positive (first) octant, but the co-ordinates of all points on the surface are not triples.

The parabolic cylindrical surface with elements parallel to the z-axis determined by the equation $x^2=2y+1$, referred to the same axes, will intersect the cone in a curve that will contain, in the first octant, all the points whose co-ordinates are triples in the set for which $a=1$. The planes parallel to the xy-plane corresponding to $z=5, 13, 25, \cdots$, will intersect the cone in circles which will intersect the intersection of the cone with the parabolic cylindrical surface at the triple points.

By taking all the positive integral values of a and constructing the corresponding parabolic cylindrical surfaces and circles, the location of all the discrete "triple points" could be determined, or, of course, they could be plotted directly.

Though not included in the scope of the title of this article, it is of interest to note that an unusual trigonometric relationship exists between the triples (3, 4; 5) and (7, 24; 25). The sine of the acute base angle opposite the longer leg of the second triangle is equal to the sine of twice the corresponding angle of the first triangle. Let the base angle of the first triangle opposite the side equal to 4 be θ. Let the base angle of the second triangle opposite the side equal to 24 be ϕ. Then, $\cos\theta=\frac{3}{5}$, $\sin\theta=\frac{4}{5}$, $\sin 2\theta=2\sin\theta\cos\theta=2\cdot\frac{4}{5}\cdot\frac{3}{5}=24/25$. But $\sin\phi=24/25$. Therefore, $\sin 2\theta=\sin\phi$. But, $2\theta\neq\phi$ as $\sin(180°-2\theta)$ also equals $\sin 2\theta$, and the angle $180°-2\theta$ must be used since 2θ is greater than 90°.

This pair of triples seems to be the only pair of primitive triples that have the stated relationship between the functions of the corresponding angles. Since $\sin 2\theta=2\sin\theta\cos\theta$ and since the values of $\cos\theta$ and $\sin\theta$ will always have the same denominator, the denominator of the value of $\sin 2\theta$ must be the square of the denominators of $\cos\theta$ or $\sin\theta$. Based on this necessary condition, it is conjectured that the above pair of triples is the only pair. Perhaps some reader can prove this or find others. It is evident that the same relationship holds for the pairs of nonprimitive triples: $(3a, 4a; 5a)$ and $(7a, 24a; 25a)$. Hence, there are an infinite number of nonprimitive triples having this relationship between their angles. The writer is not aware that this relationship has previously been noted.

There are many interesting aspects of Pythagorean triples. An extensive bibliography on the subject may be found in THE MATHEMATICS TEACHER, XLIV, No. 8, 586.

> "Mathematics is a type of thought which seems ingrained in the human mind, which manifests itself to some extent with even the primitive races, and which is developed to a high degree with the growth of civilization. . . . A type of thought, a body of results, so essentially characteristic of the human mind, so little influenced by environment, so uniformly present in every civilization, is one of which no well-informed mind today can be ignorant."—*J. W. A. Young,* The Teaching of Mathematics, *London, 1907, p. 14.*

The mathematics market

EDWIN P. MARTIN, *Fort Hays Kansas State College, Fort Hays, Kansas.*

An ecologist and teacher tells of the need for mathematical training for biologists.

MARKET RESEARCH is currently an extremely active and well-publicized field. I am not prepared to tell about the properties of the market for mathematics nor to urge more obnoxious results of market research, such as creation of demand. I am prepared, however, to tell about my own needs in mathematics.

It seems to me that I can claim to be a member of three different classes of consumers of mathematical knowledge. First, as a parent interested in the education of my children, I am interested in the mathematical needs of each educated person. Second, as a teacher of biological science to undergraduate students, I am concerned with the mathematical tools needed to study biology efficiently. Finally, as a practicing ecologist, I am aware of my own need for mathematical tools for analysis and description of data.

If these remarks so far imply that I am interested in the teaching of mathematics, the implication is correct. However, from all three points of view indicated, I am also interested in mathematical researches. In my own professional lifetime I have seen several new mathematical techniques developed which have been useful. Obvious examples are from the field of nonparametric statistics. Such new techniques depend on continued additions to our store of mathematical knowledge; thus, albeit indirectly, I am also a consumer of the products of mathematical scholarship.

Let me talk about my children first. One might think that the matters mathematically pertinent to their education are too elementary to be worth much comment. I hope to convince you otherwise. Of the four points I want to make concerning mathematics for everyone only one is elementary in nature, although all of them must be elementary in approach.

Elementary arithmetic is, of course, the first sort of mathematical knowledge that everyone needs. I have heard that one mathematics teacher tells his beginning classes that there are four and only four primary mathematical operations. These are the operations that should be mastered by everyone early in his academic career. Not only should the operations themselves be grasped, but special attention should be given to the rules governing their use. Few of us try to add apples and oranges, but more subtle analogous mistakes are not rare.

A second aspect of mathematics which I want my children to learn is its aptness for description. Communication of knowledge from one person to another is an extremely difficult business; to the extent that the information can be put in mathematical terms the problem is simplified. I doubt whether everything worth knowing can be described quantitatively, but certainly many things of importance can. I want my children to acquire the habit and the techniques of such description. I hope they will not use the skill to befuddle rather than to enlighten.

The third concept with which I am concerned is the artificial nature of mathematics. Not only should everyone know that mathematics per se has nothing to do with objects, but they should also see that this is a virtue. Without becoming too technical, I think everyone can learn that mathematical statements are of only two basic types; either they are axiomatic and established by definition and convention

or they are "if-then" statements. Once this view is comprehended, both the degree of certainty in mathematical conclusions and the enormous range of applicability of mathematical techniques becomes clear, as well as the limits of their usefulness.

Finally, I think that everyone should know that mathematics, as used in the sciences, provides a good basis for probability statements about future events. While the actual mathematical treatment of most scientific matter is not a part of everyone's education, an understanding of how, and for what, science uses mathematics is, in my opinion, essential.

In summary, I feel that my children should be taught mathematics for the same type of reasons that they should be taught music. Mathematics appreciation is a fair label for what I want.

Now, let me shift to my second role, that of a college teacher of biology. In addition to the appreciation of mathematics, what do majors in biology require? College biology students should first of all understand algebra and be skillful in its use. In addition to the skills required to solve problems, I want to emphasize the need for the ability to formulate problems in algebraic form. Especially, in this connection, these students of science need to be able to identify what is given and what is to be calculated. It is a commonplace for science students, competent with algebraic operations, to be unable to solve problems in science courses because they cannot state the problem algebraically.

Secondly, my biology students need the concept of functions when they begin their studies, and most of them do not have this concept. Quantitative statements of relationships among variables are a basic part of biology. In my opinion, they should be introduced even into high school biology, and they are indispensable to the study of even elementary college biology. While much of the mathematical knowledge of functions is beyond high school and beginning college biologists, the study of relationships of variables with only elementary mathematics has been demonstrated recently in the early morning physics class on television. The major share of the responsibility for introducing the concept of functions into elementary biology, of course, rests with the biology teacher, but it is necessary that the mathematics courses taken by prospective biology teachers give them the necessary knowledge.

The third thing I wish my students understood when they begin to study college biology is the descriptive uses of mathematics. First of all, this means an ability to at least read and understand descriptive statistics, even though they are unable to use statistical techniques. Such quantities as means, variance, correlation coefficient, confidence limits, and standard error are used so frequently in biological writing that ignorance of them almost amounts to illiteracy. A second sort of description which is important to biologists is that accomplished by graphs. A college biology student must be able to read graphs and should understand, for instance, what slopes mean in biological phenomena. After all, rates of change are ubiquitous in the study of biology, and their simpler mathematical expressions are almost indispensable. Finally, somehow these embryo biologists must grasp the fundamental idea that mathematical description is one kind of abstraction and has all the advantages and disadvantages of abstract thought. Specifically, they should learn that relationships among dissimilar phenomena are best, and sometimes only, studied via abstraction, and that confusion between an abstraction from phenomena and the phenomena per se must be avoided.

And finally, what sort of mathematical knowledge is needed by a professional biologist? Of course, the detailed mathematical needs of biologists vary as much as the problems they study. Certain kinds of mathematics, however, are used in almost any sort of research. First, perhaps, we

should mention statistics. Not only is a statistical vocabulary needed to read current literature, but some skill with statistical techniques is needed both in planning investigations and in analyzing results. Included in statistical knowledge must be an understanding of the assumptions which data must satisfy before the use of statistical analysis is justified. Due to the great current emphasis on quantitative biology, statistics are abused about as often as they are used.

Most biologists also find occasion to use analytic geometry and sometimes the calculus. Models are widely used, both to establish hypotheses for testing and for more effective communication of results. Often, such models can be deduced mathematically from physical and chemical laws. In more complicated cases, a sensible biologist will beg or buy professional mathematical assistance, but the simpler applications should be within our reach.

As a conclusion to these remarks, let me describe a specific and perhaps surprising use of mathematics which I have found very helpful both in teaching and in studying biology. Many biological processes are affected by several factors, some accelerators and some decelerators. Examples include the rate of heartbeat, rate of change in densities of population, and rate of learning. If each of the factors influencing these rates is measured and treated as a vector, the results of their interaction can be both studied and explained with surprising simplicity.

The example shows that mathematical knowledge proves helpful to biologists in unexpected ways. No doubt any increase in such knowledge is good. I have tried to show, however, what the minimum requirements are for three sections of the mathematics market. Let me also urge that, somehow, the power and elegance of mathematics become a part of both our product and our advertising. Perhaps "every man a mathematician" is a goal neither possible nor desirable, but I hope we can, before long, produce a society in which every man is mathematically literate.

What's new?

BOOKS

SECONDARY

High School Mathematics, Units 1 to 4, University of Illinois Committee on School Mathematics. Urbana, Illinois: University of Illinois Press, 1959. Paper, 618 pp., $2.25. Teacher's edition, $6.00.

MISCELLANEOUS

A Concrete Approach to Abstract Algebra, W. W. Sawyer. San Francisco: W. H. Freeman and Company, 1959. Paper, 233 pp., $1.25.

From Euclid to Eddington, A Study of Conceptions of the External World, Sir Edmund Whittaker. New York: Dover Publications, Inc., 1959. Paper, ix+212 pp., $1.35.

On Mathematics and Mathematicians, Robert Edouard Moritz. New York: Dover Publications, Inc., 1959. Paper, v+410 pp., $1.95.

Philosophy and the Physicists, L. Susan Stebbing. New York: Dover Publications, Inc., 1959. Paper, xvi+295 pp., $1.65.

The Philosophy of Space and Time, Hans Reichenbach. New York: Dover Publications, Inc., 1959. Paper, xvi+295 pp., $2.00.

BOOKLETS

Problem Situations in Teaching, Gwynn A. Greene. New York: Bureau of Publications, Teachers College, Columbia University, 1959. Paper, vii+69 pp., $1.25.

A "long" method of factoring quadratic trinomials

ROBERT C. McLEAN, JR., *Los Angeles City Schools, Los Angeles, California.*

An approach to factoring trinomials that emphasizes the distributive law.

WHEN THE ADVOCATES of modernizing the high school algebra curriculum bring up the matter of factoring, they insist that it should be based upon the use of the distributive law of multiplication. In the case of common monomial factors, such application is obvious; but in the case of quadratic trinomials, it would not seem to be so. In trying to develop a demonstration for teacher-groups to illustrate the truth of the assertion of the authorities, the following method of analyzing a quadratic trinomial for factoring was developed. Teachers might find it helpful in that it breaks down the process into smaller steps than does the usual process of "guessing" while trying to fill in four empty places in two sets of parentheses.

The letters a, b, c, d represent integers, and the coefficient of x^2 is not zero.

First, whenever two binomials are multiplied, the product is a quadrinomial until like terms are combined. That is, $(a+b)(c+d)$ is $ac+ad+bc+bd$. Furthermore, this product is obtained by applying the distributive law twice, for example: $(a+b)c+(a+b)d$ would be the first application, then it would be applied again to each part of this expression.

Second, whenever two binomials of the form $(ax+b)(cx+d)$ are multiplied, the product is of the form $acx^2+(ad+bc)x+bd$. The coefficient of x is made up of two parts whose sum is the required coefficient. Furthermore, their product equals the product of the coefficient of x^2 and the constant term. These products, $adbc$ and $acbd$, are equal, for the multiplication of integers is commutative.

With this analysis of how quadratic trinomials are formed by multiplication, let us try to reverse the process and factor some examples.

(1) The coefficient of x^2 is 1, as in $x^2-2x-15$.

Set down the first and last terms with some room between them, for we shall make a quadrinomial of our trinomial by inserting two appropriately chosen middle terms:

$$x^2 \qquad\qquad -15.$$

For our middle terms we set the requirements that their sum be equal to the original middle term, and that the product of the coefficients equal the product of 1 and -15. (See the second remark above.) With a little patience one finds these to be $3x$ and $-5x$. Insert these in the expression, getting:

$$x^2+3x-5x-15.$$

Factor this expression by grouping in this manner:

$$x(x+3)-5(x+3).$$

This is the first application of the distributive law. Factor again, taking out the common binomial factor:

$$(x-5)(x+3).$$

This is the second application of the distributive law and gives the required pair of binomial factors. If the middle terms are switched, the same factors are obtained, except they are in the other order.

(2) The coefficient of x^2 is other than 1, as in $8x^2-22x+15$.

Set down the first and last terms with some room between them:

$$8x^2 \qquad\qquad +15.$$

To fill this space, find two integers whose sum is -22 and whose product is (8)(15) or 120. Numbers meeting this requirement are -10 and -12. Using these we have:

$$8x^2-10x-12x+15.$$

Factor this expression by grouping in this manner:

$$2x(4x-5)-3(4x-5).$$

Factor out the common binomial factor:

$$(2x-3)(4x-5).$$

These are the required factors.

Considering this as a method of factoring, it takes four steps. Thus it is a long method compared to the usual system of setting down the factors directly. For a student who has trouble with this type of factoring, it might prove advantageous, in that he is not asked to do everything at once. In particular, he can decide about the appropriate coefficients before he has to decide where to put the negative terms in the binomial factors.

(3) Just to make things complete, here is an example of a quadratic binomial factored in this style. This is not recommended, but it simply shows that it can be done.

Factor:

$$\begin{aligned} & x^2-4 \\ &= x^2-2x+2x-4 \\ &= x(x-2)+2(x-2) \\ &= (x+2)(x-2). \end{aligned}$$

Letter to the editor

Dear Editor:

I have just gotten around to reading the February 1959 issue of THE MATHEMATICS TEACHER.

On page 150 in "Letters to the editor," Professor James H. Keller writes of a problem and its proof worked out by a freshman student of his. I will comment briefly on this problem.

1. The result obtained, namely c/a, of course, suggests the product of the roots of $ax^2+bx+c=0$. If we label the origin C, the point $(r_1, 0)$, B, the point $(r_2, 0)$, A, and the turning point of the parabola $y=ax^2+bx+c$ (there is no loss of generality in assuming a positive) by D, the configuration obtained is the familiar "ambiguous" s.s.a. case of trigonometry.

Applying the Cosine Law to each of the triangles CBD and CAD, we have

$$a^2=b^2+c^2+2bc\cos A$$

and

$$a^2=b_1^2+c^2-2b_1c\cos A.$$

Therefore,

$$\begin{aligned} a^2-c^2 &= b^2+2bc\cos A \\ &= b(b+2c\cos A) \\ &= r_1r_2 \end{aligned}$$

and

$$\begin{aligned} a^2-c^2 &= b_1^2-2b_1c\cos A \\ &= b_1(b_1-2c\cos A) \\ &= r_2r_1. \end{aligned}$$

2. The proof given by Mr. Lloyd is restricted to real roots of $ax^2+bx+c=0$.

3. A minor misprint on the third line from the end of the letter: the last term of the numerator should read $16a^2c^2$.

Sincerely yours,
CHARLES T. SALKIND
Polytechnic Institute of Brooklyn
Brooklyn, New York

• HISTORICALLY SPEAKING,—

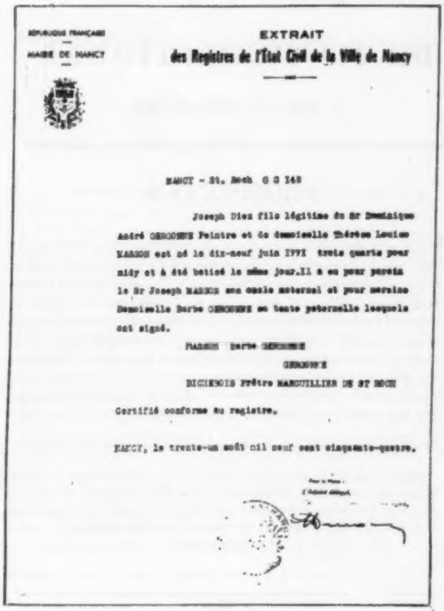
Edited by Howard Eves, University of Maine, Orono, Maine

Gergonne, founder of the Annales de Mathématiques

by Laura Guggenbuhl, Hunter College, New York, New York

Joseph Diez Gergonne was born on June 19, 1771, in the parish of St. Roch at Nancy, just a few years after this city had been transformed into a center of artistic elegance. Indeed, his father had been one of the artists who had had a part in the creation of this wonderland of lavish grace.

As the only son of Thérèse Louise Masson and Dominique André Gergonne, painter to the king, the lad seemed destined for a career in the arts. However, his father's death, when he was only twelve years old, made it necessary for him to find some more immediate means of earning his livelihood. From his earliest years in the parochial schools at Nancy, he had shown a gift for science and numbers. Thus it is no surprise to hear that by the time he was seventeen years old, he could



RÉPUBLIQUE FRANÇAISE
MAIRIE DE NANCY

EXTRAIT
des Registres de l'État Civil de la Ville de Nancy

NANCY - St. Roch G G 148

Joseph Diez fils légitime du Sr Dominique André GERGONNE Peintre et de demoiselle Thérèse Louise MASSON est né le dix-neuf juin 1771 trois quarts pour midy et a été batisé le même jour. Il a eu pour parein le Sr Joseph MASSON son oncle maternel et pour mareine Demoiselle Barbe GERGONNE sa tante paternelle lesquels ont signé.

MASSON Barbe GERGONNE
GERGONNE
RICHEBOIS Prêtre MARGUILLIER DE ST ROCH

Certifié conforme au registre.

NANCY, le trente-un août mil neuf cent cinquante-quatre.

Figure 1

Birth Certificate.
Courtesy of Mairie de Nancy.

Figure 2

Joseph Diez Gergonne, 1771–1859.
Courtesy of Public Library of Nancy.

support his mother and sister by giving private lessons in mathematics.

When his homeland was threatened by invading forces in 1792, he volunteered for service in the army. With the exception of a few months in Paris, as secretary to an uncle, he spent the next few years in military service. He became an artillery officer, and in 1795 his regiment was sent to Nîmes.

Perhaps the surroundings at Nîmes—the elaborate Jardin de Fontaines, the ancient coliseum, and the ruins of masterpieces of Roman architecture—reminded the young officer of the splendor of his birthplace. In any case, when he was appointed to the chair of mathematics at the newly established Central School at Nîmes, he chose to make his home in this part of France.

Six years later he set out on a long and eventful journey which took him to Berlin to marry a young woman he had met while in Paris. On his return, he and his bride spent several happy weeks in Nancy, visiting relatives and old friends, before proceeding to the new home in the Gard.

The years at Nîmes were successful, and Gergonne prospered in his academic pursuits. When he was forty-five years old he was named professor of astronomy at nearby Montpellier, and fourteen years later he became the sixth rector of the Academy. Always an indefatigable worker, he insisted upon continuing a limited amount of teaching along with his duties as rector. In the classroom and in administrative office he was a stern, but fair, disciplinarian, and he was always quick to speak out against any form of injustice.

ANNALES

DE

MATHÉMATIQUES

PURES ET APPLIQUÉES.

RECUEIL PÉRIODIQUE,

RÉDIGÉ

Par J. D. GERGONNE et J. E. THOMAS-LAVERNÈDE.

TOME PREMIER.

A NISMES,

DE L'IMPRIMERIE DE LA VEUVE BELLE.

Et se trouve à PARIS, chez COURCIER, Imprimeur-Libraire pour les Mathématiques, quai des Augustins, n.° 57.

1810 ET 1811.

Figure 3

Front cover, *Annales de Mathématiques*, Vol. I, 1810.
Courtesy of Bibliothèque Nationale.

ANNALES

DE MATHÉMATIQUES

PURES ET APPLIQUÉES.

PROSPECTUS.

C'EST une singularité assez digne de remarque que, tandis qu'il existe une multitude de journaux relatifs à la *Politique*, à la *Jurisprudence*, à l'*Agriculture*, au *Commerce*, aux *Sciences physiques et naturelles*, aux *Lettres* et aux *Arts*; *les Sciences exactes*, cultivées aujourd'hui si universellement et avec tant de succès, ne comptent pas encore un seul recueil périodique qui leur soit spécialement consacré (*); un recueil qui permette aux Géomètres d'établir entre eux un commerce ou, pour mieux dire, une sorte de communauté de vues et d'idées; un recueil qui leur épargne les recherches dans lesquelles ils ne s'engagent que trop souvent en pure perte, faute de savoir que déjà elles ont

(*) On ne saurait, en effet, considérer comme tels, le *Journal de l'école polytechnique*, non plus que la *Correspondance* que rédige M. Hachette: recueils très-précieux sans doute, mais qui, outre qu'ils ne paraissent qu'à des époques peu rapprochées, sont consacrés presque uniquement aux travaux d'un seul établissement.

Tom. I, n.° I, 1.er juillet 1810.

Figure 4

Statement of aims, *Annales de Mathématiques*, Vol. I, 1810.
Courtesy of Bibliothèque Nationale.

At the age of seventy-three, Gergonne retired. The years of taxing devotion to duty, scientific research, prolific publication, and the writing of thousands of letters gave way to quiet years devoted to an adoring family. The fourteen years of his retirement, marked also by many hours given over to religious reflection, were incredibly quiet. Gergonne died on April 4, 1859 and was buried at Montpellier.

It was in 1810 at Nîmes that Gergonne founded the *Annales de Mathématiques*, the first journal in France devoted to mathematics. Volumes I and II carried the name of Joseph E. Thomas-Lavernède as coeditor (see Figure 3), but from Volume III on Gergonne is listed as the sole editor.

There is ample evidence to indicate that the journal was an intensely personal mat-

CERCLE TANGENT A TROIS AUTRES. 289

GÉOMÉTRIE ANALITIQUE.

Recherche du cercle qui en touche trois autres sur un plan;

Par M. GERGONNE.

IL y a environ trois ans que l'académie de Turin voulut bien rendre public, par la voie de l'impression, un mémoire que je lui avais adressé, et où, dans le dessein de venger complètement la géométrie analitique du reproche qu'on ne lui fait que trop souvent de ne pouvoir rivaliser avec la géométrie pure, pour la construction des problèmes, j'essayais de prouver que cette géométrie analitique, convenablement maniée, offrait les solutions les plus directes, les plus élégantes et les plus simples de deux problèmes dès-long-temps célèbres, et qui passent pour difficiles : je veux parler du problème où il s'agit de *décrire un cercle qui touche trois cercles donnés* et de celui où il est question de *décrire une sphère qui touche quatre sphères données.*

J'écrivais pour des savans consommés, et je crus devoir être court; il paraît que je le fus un peu trop; plusieurs géomètres, qui eurent connaissance de mon mémoire, me firent le reproche, fondé sans doute, que le fil qui m'avait guidé n'y était pas assez apparent, et que mes calculs semblaient plutôt propres à légitimer une construction trouvée par un heureux hasard, qu'à faire découvrir cette construction. Il paraît même que, par suite de mon excessif laconisme,

Tom. VII, n.° X, 1.er avril 1817.

Figure 5

Title page of Gergonne's article on Solution of Apollonian Problem by Analytic Geometry. *Annales de Mathématiques*, Vol. VII, 1817. Courtesy of Bibliothèque Nationale.

CERCLE TANGENT A TROIS AUTRES. 193

GÉOMÉTRIE ÉLÉMENTAIRE.

Sur la construction du cercle tangent à trois cercles donnés;

Par un ABONNÉ.

Au Rédacteur des *Annales*;

MONSIEUR,

EN examinant avec attention l'ingénieuse théorie par laquelle M. le professeur Durrande, au commencement du XI.e volume des *Annales*, est parvenu à démontrer géométriquement l'élégante construction que vous avez déduite de l'analise algébrique, à la page 302 du VII.e volume du même recueil, pour la détermination du cercle qui touche à la fois trois cercles donnés, il m'a paru que cette théorie était susceptible de simplification assez notables; qu'elle pourrait être rendue indépendante de tout calcul, et même de la considération des proportions, et qu'elle devenait ainsi, en quelque sorte, le résultat d'une pure intuition. J'ai l'honneur de vous transmettre le résultat de mes réflexions sur ce sujet, dont vous ferez l'usage que vous jugerez convenable.

J'admets uniquement les principes connus sur les *pôles* et *polaires* et sur les *axes radicaux*, principes que l'on peut démontrer soit à la manière de Monge, soit comme l'a fait M. Durrande, soit de toute autre manière; et je les rappelle en ces termes:

1. Les sommets de tous les angles circonscrits à un même cercle,

Figure 6

Title page of an article by an anonymous subscriber (Gergonne) on Construction of a Circle Tangent to Three Given Circles Using Theory of Poles and Polars, etc. *Annales de Mathématiques*, Vol. XIII, 1822. Courtesy of Bibliothèque Nationale.

ter with Gergonne. He contributed heavily out of his own pocket to insure its continued appearance, and he was one of its most prolific authors. Here at last his artistic inheritance found an outlet, for he drew all the diagrams himself. Sometimes, behind the thinly-veiled disguise of an anonymous subscriber, he sent letters to the editor, and proposed problems for solution. As editor, he would then promptly resolve these matters.

No doubt these practices are time-honored editorial customs. But everyone will agree that in the *Annales*, Gergonne established in form and content a set of exceptionally high standards for mathematical journalism. New symbols and new terms to enrich mathematical literature are found here for the first time. The

journal, which met with instant approval, became a model for many another editor. Such famous mathematicians as Cauchy, Poncelet, Brianchon, Steiner, Plücker, Crelle, Poisson, Ampère, Chasles, and Liouville sent articles for publication.

In 1814, Gergonne's first masterpiece, a discussion and solution of the problem of finding a circle tangent to three given circles, was published. Strangely enough, his paper made its first appearance not in the *Annales*, but in the *Mémoires of the Royal Academy of Sciences of Turin*. The treatment of the problem was purely analytic; it was based upon a set of co-ordinate axes and a set of simultaneous equations involving the co-ordinates of the points of contact of the desired circle with the three given circles.

Three years later Gergonne wrote a paper on the same subject from substantially the same point of view in the *Annales*, and in succeeding years the problem was the subject of many additional articles. Subsequent developments provided a neat solution based on the theory of poles and polars, radical center, and centers of similitude. These latter articles were accompanied by a ruler and compass construction. Gergonne's treatment of the Apollonian problem has become a classic, and is now regarded as the most elegant of all solutions.

Although Gergonne went to Montpellier in 1816, the *Annales* continued to be published at Nîmes. Around 1826 the journal became a battleground for a feud which raged between Poncelet and Gergonne regarding the principle of duality. Each claimed priority in its discovery. A succinct summary is given in an inconspicuous footnote on page 148 of Volume XVIII. Gergonne says briefly:

"Voici du nouveau pour moi. Je persiste pourtant à croire, n'en déplaise à M. Poncelet, que mes quelques articles à deux

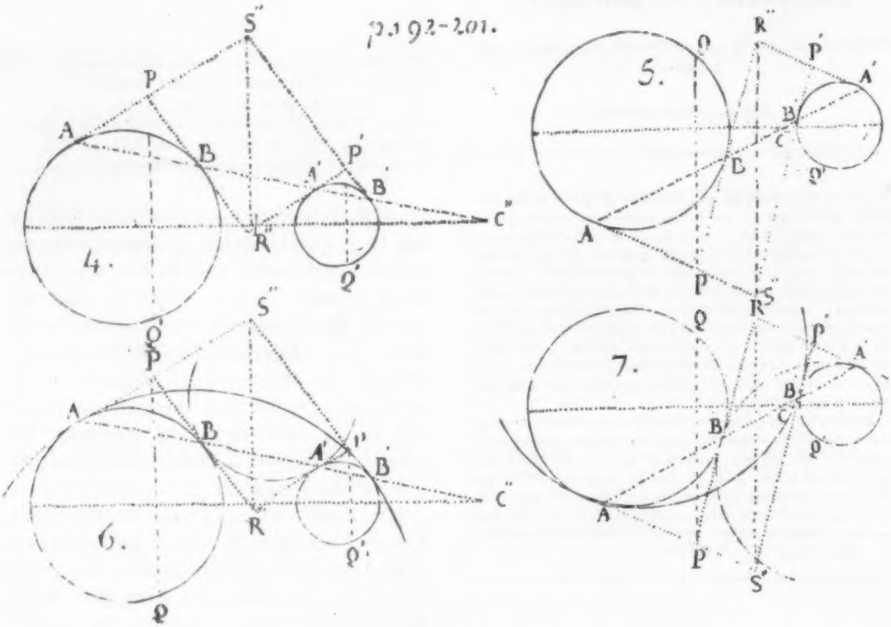


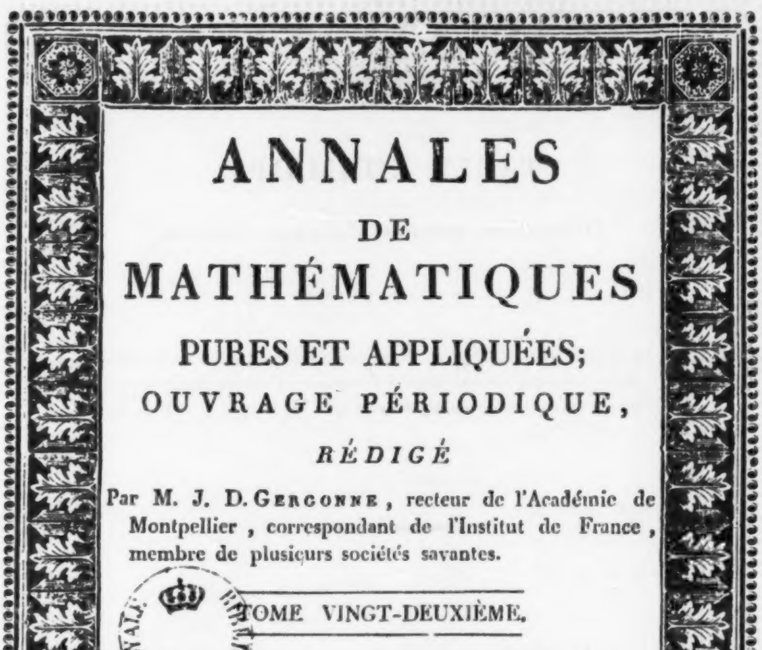
Figure 7

Diagram to illustrate construction of a circle tangent to three given circles. *Annales de Mathématiques*, Vol. XIII, 1822. Courtesy of Bibliothèque Nationale.

colonnes ont plus efficacement servi la cause de la dualité que ne l'ont fait les 400 pages de son ouvrage."

Today the system of writing corresponding statements in two parallel columns is universally used in books on projective geometry. Gergonne regarded the principle of duality as one of his most interesting discoveries, and it was the subject of his last publication.

It has been in vain that we have searched for a reference in Gergonne's articles in the *Annales* to the Gergonne point of a triangle, namely, the point of intersection of the lines from the vertices of a triangle to the points of contact of the inscribed circle with the opposite sides of the triangle. This point seems to have been discovered by another, and it is not clear how it came to be called the Gergonne point.



ANNALES
DE
MATHÉMATIQUES
PURES ET APPLIQUÉES;
OUVRAGE PÉRIODIQUE,
RÉDIGÉ
Par M. J. D. GERGONNE, recteur de l'Académie de Montpellier, correspondant de l'Institut de France, membre de plusieurs sociétés savantes.

TOME VINGT-DEUXIÈME.

N.° II. — AOUT 1831.

A NISMES, DE L'IMPRIMERIE DE P. DURAND-BELLE;
Et se trouve A PARIS,
Chez BACHELIER, gendre COURCIER, Libraire, pour les Mathématiques, Quai des Augustins, n.° 55.

Figure 8

Front cover of last volume of *Annales de Mathématiques*, Vol. XXII, 1831. Courtesy of Bibliothèque Nationale.

On the back cover of Volume XXII, 1831 (see Figure 9), one can read an apologetic little note saying that Gergonne's new duties as Rector, in addition to his teaching duties, made it impossible for him to continue the publication of his journal. And thus the *Annales de Mathématiques* came to an end, after having appeared with unfailing regularity and punctuality every month for twenty-two years.

In 1836, the *Journal de Mathématiques pures et appliquées* of Liouville was founded and described as a successor to the *Annales*. The *Nouvelles Annales de Mathématiques*, claiming no similarity or relationship whatsoever to Gergonne's *Annales*, was founded in 1842. Yet it is in the *Nouvelles Annales de Mathématiques* that one may read a contemporary obituary notice for Gergonne.

TABLE SOMMAIRE

Des matières contenues dans cette livraison.

ANALYSE ALGÉBRIQUE. Nouveaux développemens sur les procédés élémentaires qui peuvent servir à déterminer les valeurs maximums *et* minimums *des fonctions algébriques ;* par M. Bary. Pag. 37

N. B. La ponctualité que nous avons apportée pendant vingt ans dans la publication de nos livraisons doit assez avertir nos lecteurs que, si elles ont éprouvé, dans ces derniers temps, un retard assez notable, il n'a pas tenu à nous qu'il en fût autrement. Chargés depuis dix huit mois de la direction de l'enseignement dans quatre départemens, il nous a fallu faire trêve à nos douces et paisibles occupations pour nous livrer à de fastidieux détails administratifs. Présentement que la machine est à peu près montée, nous ne négligerons rien pour nous remettre bientôt au courant.

Figure 9

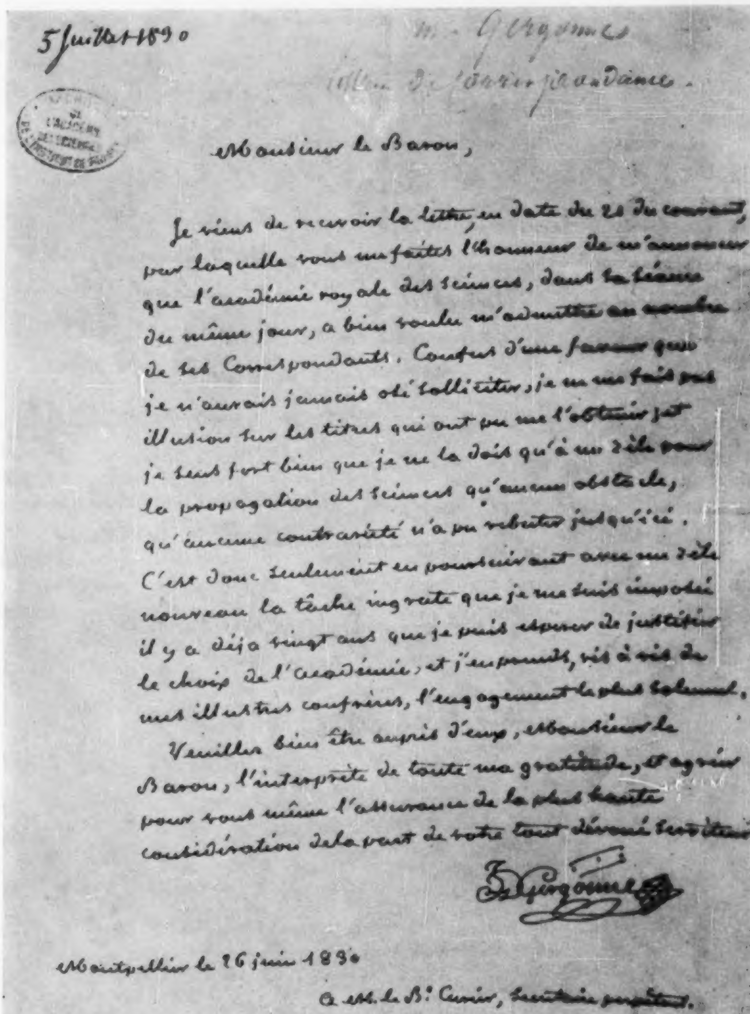
Back cover of last volume of *Annales de Mathématiques*, Vol. XXII, 1831. Courtesy of Bibliothèque Nationale.

A letter (Fig. 11) preserved at the Public Library at Nancy bids Godron, a later Rector of the Academy at Montpellièr, to come to Gergonne's home to receive the cross of the Legion of Honor. It is a charming document, particularly so because it is a voice out of the years of quiet retirement.

The chief sources of information for this article are:

Bouisson, F., "Notice biographique sur Joseph-Diez Gergonne, ancien recteur de l'Académie de Montpellier," *Montpellier Médical*, 3 (1859), No. 2, p. 176. University of Montpellier.

Gervais, P., *Discours prononcé aux*



5 Juillet 1830

M. Gergonne
Mem. de Correspondance.

Monsieur le Baron,

Je viens de recevoir la lettre, en date du 28 du courant,
par laquelle vous me faites l'honneur de m'annoncer
que l'académie royale des sciences, dans la séance
du même jour, a bien voulu m'admettre au nombre
de ses Correspondants. Confus d'une faveur que
je n'aurais jamais osé solliciter, je ne me fais pas
illusion sur les titres qui ont pu me l'obtenir; et
je sens fort bien que je ne la dois qu'à un zèle pour
la propagation des sciences qu'aucun obstacle,
qu'aucune contrariété n'a pu rebuter jusqu'ici.
C'est donc seulement en poursuivant avec un zèle
nouveau la tâche ingrate que je me suis imposée
il y a déja vingt ans que je puis espérer de justifier
le choix de l'académie, et j'en prends, vis à vis de
mes illustres confrères, l'engagement le plus solennel.

Veuillez bien être auprès d'eux, Monsieur le
Baron, l'interprète de toute ma gratitude, et agréez
pour vous même l'assurance de la plus haute
considération de la part de votre tout dévoué serviteur

J D Gergonne

Montpellier le 26 juin 1830

À M. le Bn Cuvier, Secrétaire perpétuel.

Figure 10

Letter written by Gergonne when he was named a member (correspondant) of the Institut, 1830.
Courtesy of l'Académie des Sciences de l'Institut de France.

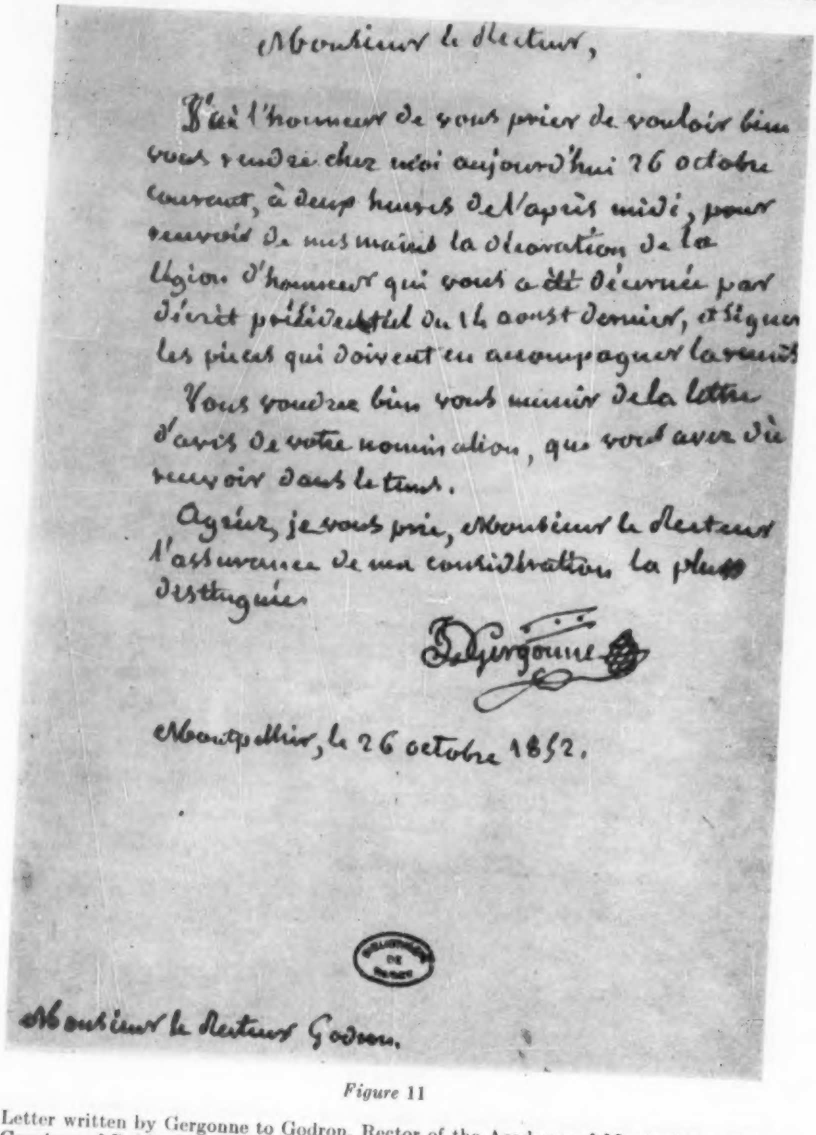
funérailles de M. Gergonne, Professeur honoraire. Montpellier: 1859. University of Montpellier.

LAFON, A., *Gergonne, sa vie et ses travaux.* Nancy: 1861. Bibliothèque Nationale.

TERQUEM, O., "Gergonne (Obituary Notice)," *Nouvelles Annales de Mathématiques,* 18 (1859), p. 40. Bibliothèque Nationale.

APPENDIX

In this appendix we give a brief discussion of the construction of a circle tangent



Monsieur le Recteur,

J'ai l'honneur de vous prier de vouloir bien vous rendre chez moi aujourd'hui 26 octobre courant, à deux heures de l'après midi, pour recevoir de mes mains la décoration de la Légion d'honneur qui vous a été décernée par décret présidentiel du 14 août dernier, et signer les pièces qui doivent en accompagner la remise.

Vous voudrez bien vous munir de la lettre d'avis de votre nomination, que vous avez dû recevoir dans le temps.

Agréez, je vous prie, Monsieur le Recteur l'assurance de ma considération la plus distinguée.

J. D. Gergonne

Montpellier, le 26 octobre 1852.

Monsieur le Recteur Godron.

Figure 11

Letter written by Gergonne to Godron, Rector of the Academy of Montpellier, 1852. Courtesy of Public Library of Nancy.

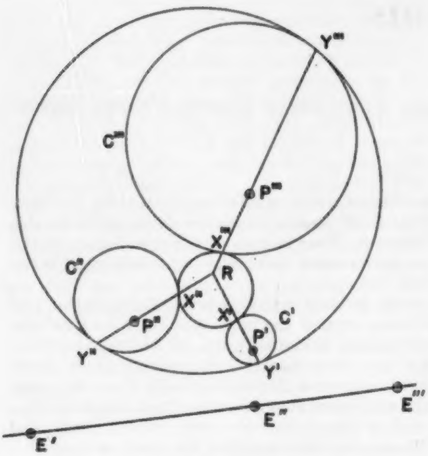


Figure 12

Diagram to illustrate appendix.
Construction of a circle tangent to three given circles.

to three given circles, according to Gergonne's classic solution of the Apollonian problem.

At the outset one must realize that the number of circles tangent to three given circles vary from zero to eight, depending upon the relative positions of the given circles. The following paragraph shows how two such circles may be drawn.

In Figure 12, let c', c'', and c''' be the three given circles. Let the external centers of similitude of these circles, taken in pairs, be E', E'', and E''', E' being that of circles c'' and c''', etc. Since the external centers of similitude of three circles, taken in pairs, are collinear, let l represent the line on which E', E'', and E''' are situated. Let the points P', P'', and P''' be the poles of the line l with respect to the circles c', c'', and c'''. Call the radical center of the three given circles R. Let the lines RP', RP'', and RP''' cut the circles c', c'', and c''' in the pairs of points X' and Y', X'' and Y'', and X''' and Y''' respectively. The circles through the points $X'X''X'''$ and $Y'Y''Y'''$ are two circles tangent to the three given circles. A proof is readily available in most textbooks on modern geometry.

To find additional circles of the complete solution, one may proceed as follows. Since each external center of similitude is collinear with two internal centers of similitude of three given circles taken in pairs, there are three lines l_1, l_2, l_3 analogous to the line l in the preceding paragraph. The poles of the lines l_1, l_2, l_3 are connected with the radical center R to find the points of contact of the other three pairs of tangent circles. If the points of contact fail to materialize, because of the relative positions of the three given circles, the corresponding tangent circles do not exist.

"Mathematics is a living, growing subject. The vitality and vigor of present-day mathematical research quickly dispels any notion that mathematics is a subject long since embalmed in textbooks. Mathematics today is in many respects an entirely different discipline from what it was at the turn of the century. New developments have been extensive; new concepts have been revolutionary. The sheer bulk of current mathematical development is staggering."—"Program for college preparatory mathematics," *Report of the Commission on Mathematics, College Entrance Examination Board, New York, 1959, p. 1.*

Reviews and evaluations

Edited by Kenneth B. Henderson, University of Illinois, Urbana, Illinois

BOOKS

Fundamentals of Freshman Mathematics, C. B. Allendoerfer and C. O. Oakley (New York: McGraw-Hill Book Company, Inc., 1959). Cloth, xiii+475 pp., $6.50.

This book, written in a vein similar to that of *Principles of Mathematics* (McGraw-Hill Book Company, Inc., 1955) is intended to bridge the gap between intermediate algebra and analytic geometry and calculus. Unlike its predecessor, it will impress the teacher of traditional mathematics with the relative familiarity of much of its contents. It is contemporary in flavor but succeeds admirably in reassuring teachers that much of their own undergraduate work in mathematics still possesses intrinsic merit.

The authors have geared their text to the following levels:

1. Twelfth-grade mathematics in high school—covers materials considered as necessary for the College Entrance Examination Board examinations in elementary analysis.
2. College Algebra—roughly the first half of the book deals with materials usually covered by such a course.
3. College Algebra and Trigonometry—enough trigonometry is included along with the algebra to cover adequately these requirements.
4. Unified Freshman Course—the book provides enough material for a year course in mathematics for those students who enter college with some knowledge of intermediate algebra and who wish to proceed to calculus in their sophomore year.

Much of the mathematics in the book reflects the traditional content of a course in college algebra. The approach to the teaching of this material, however, is in line with the present emphasis on mathematical structure. The section on trigonometry is carried along on two levels: one, the customary trigonometry of angles and the other, the trigonometric functions of real numbers. There are adequate introductions to analytic geometry and calculus. The book contains a tremendous number of problems.

For me, the chief strength of the book lies in the fact that it will appeal both to the experienced teacher of traditional mathematics and to the neophyte. Too many texts these days lose a potential audience by the shock value of their contents. A good segment of the teaching brotherhood today is alive to the need for curricular reform in mathematics all along the line. This book makes a commendable move in this direction. That it does so without the use of the sledge-hammer technique is praiseworthy in itself.

As in their previous book, Allendoerfer and Oakley express the hope that their book is relatively free of errors, but each blames the other for any that may be discovered. How much more cause for altercation must there be when the authors of that other fine book *Finite Mathematical Structures*—Kemeny, Mirkil, Snell, and Thompson—get together for post mortems!—*Ernest R. Ranucci, Newark State College, Union, New Jersey.*

A Guide to the Use and Procurement of Teaching Aids for Mathematics, Emil J. Berger and Donovan A. Johnson (Washington, D. C.: NCTM, 1959). Paper, v+41 pp., 75¢.

This guide is a most useful and valuable publication for mathematics teachers in elementary and secondary schools. It contains much desired information on "Teaching Aids in Mathematics." Aids of one kind or another have always been used in mathematics instruction. Many teachers make effective use of the right kinds of teaching aids, or learning aids, to bring meaning and understanding of abstract concepts to their pupils. Other teachers seem not to be aware of the availability of the many aids which help "to promote learning." To all mathematics teachers this guide should be of great help.

The book (or "report") is divided into two parts: Part I deals with such topics as the function, use, and criteria for selection of teaching aids for mathematics; Part II is "deviated to a description of various types of aids, their source of supply, recommendations regarding appropriateness of specific items, and how much to spend for them."

In Part I, the definition of "teaching aids" is clear and definite. The authors state that "teaching aids are all materials, equipment, and devices which can be used to make teaching effective." In other words, "a teaching aid may be thought of as a multidimensional technique of teaching which makes use of all sensory perception to promote learning." Having given their definition, the authors classify aids into ten categories, showing a variety of kinds. This should relieve the minds of some teachers who may have thought that "aids" were just "manipulative gadgets."

The section on "Function of Teaching Aids" is well explained. "Function" is described as a "teacher's reaching out for something" to estab-

lish a meaning, to make clear a concept, or to clinch an idea. Thus, the main function of teaching aids is that they "serve a useful purpose in promoting understanding of concepts and principles." The authors defend their ideas on the sixteen purposes or goals for which teaching aids can be used by saying that "the experience of successful teachers and the results of research indicate that teaching aids can be used to serve the purposes described." The caution—"The mere use of teaching aids does not insure that the goals described will be attained"—is good. Appropriate teaching aids must be used at the right time and in the right way to be effective, for they "are only part of the learning situation and are always subordinate to aims and methods."

"How to Use Teaching Aids Effectively" is an excellent outline of procedure, or guiding principles, not only for beginning teachers but also for the experienced teachers who wish to use aids.

The "Criteria for Selection of Materials and Equipment" are such as should be used for selecting any teaching material for mathematics, as "Items should make a pertinent and significant contribution to the objectives of mathematics education."

In Part II, "The Procurement of Teaching Aids" contains an extensive and careful listing of aids classified in the order in which materials might be acquired. Since acquiring materials depends largely on the type of teaching to be done, both teachers and procurement officers are urged to read references and send for catalogs before making final decisions. This listing with estimated maximum costs and source of supply includes: Elementary Arithmetic Devices and Models; Models, Instruments, and Devices for Grades 7–12; Science Apparatus; Toys; Commercial Producers of Materials for Mathematics Teaching; Games.

The authors have listed materials suitable for purchase under the National Defense Education Act of 1958 and have included two proposed floor plans and furnishings needed for the mathematics classroom. These suggestions are sensible.

Comprehensive bibliographies of books, periodicals, pamphlets, charts, and reference materials for libraries are rich sources for teachers.

The guide is well written and easy to read. In the opinion of this reviewer, every mathematics teacher and administrator of elementary and secondary schools should have this guide or have access to it.—*Catherine A. V. Lyons, University School, Pittsburgh, Pennsylvania.*

Introduction to Analysis, Norman B. Haaser, Joseph P. LaSalle, and Joseph A. Sullivan (Boston: Ginn and Company, 1959). Cloth, xiv+688 pp.+xxxi, $8.50.

This book has the general framework of most texts on the calculus with analytic geometry, though here it is developed almost exclusively with modern ideas, tools, and symbolism. It is probable that a confirmed traditionalist will experience some degree of shock in scanning its pages for the first time.

This is Volume I of a projected two-volume set growing out of some eight years of experimentation by the authors in developing an undergraduate mathematics program for science and engineering (and some liberal arts) students at Notre Dame. Volume I is designed to be fundamental, so that the extensions of Volume II should not seem to be an entirely new subject area.

The authors state that "the purpose of this book is to present elementary analysis both as mathematics and as an instrument of science and to do this in the spirit and in the light of contemporary mathematics."

A careful reading of the book leaves little doubt that the authors have gone all out to achieve the second part of their avowed purpose.

The text begins with the axioms for the real number system and takes off with a vengeance into plane analytic geometry, functions, graphing of equations, analytic trigonometry, induction, limits and derivatives with applications, and integration with applications. Those portions of conventional college algebra and trigonometry which are considered of prime need are woven into the pattern as required.

Problems are given at the end of each section, and there is also a graded list of general problems. This appears to be a sound idea. Some answers are provided in the back of the book. But why don't the authors number their problem and answer lists so that they can be readily identified and located?

It is delightful and refreshing to see the much-neglected and underrated topic of inequalities receive most of its rightful attention. Analysis is really the theory of inequalities, and recognition of this obviously forgotten fact gives added strength to this volume.

Another happy feature is the early treatment of vectors, so that the student is primed well in advance for a more sophisticated exploration of solid analytic geometry—evidently to appear in Volume II. The treatment, however, is so deeply couched in modern symbolism that it might well tempt a freshman engineer into consideration of another academic major.

Another strong feature is the early presentation of mathematical induction—so often given scant treatment in standard works. This is probably the most traditional of all the fifteen chapters. The accompanying exercises are quite ordinary, avoiding any application to geometry. At least, no reference is made to the tiresome tumbling of dominoes suitably set in a row!

By contrast, the handling of function (Chapter 3) is anything but typical—so much so that unless the student's high school training was quite thorough at this point (it frequently is not) he would likely finish Chapter 3 loaded with set notation but having only a fuzzy notion of this vital mathematical idea. Surely this could be improved by starting out with a more conventional treatment, followed by what is given.

The presentation of limits, using the epsilon-

delta definition, is thorough, with good exercises, and worthy of praise. In the same chapter are given techniques of how to differentiate composite functions, which is all well and good, but the graphical explanation on page 350 (Figure 18) as to how this is done might well bathe a beginner in cold sweat. Add to this the idea of "points of accumulation," and you have a rugged Chapter 8.

The chapter on analytic trigonometry is possibly the weakest, which is unfortunate since so many high school trigonometry courses dwell largely on solution of triangles and put insufficient stress on trigonometric equations and inverse functions. However, a fine job is done on the graphing of trigonometric functions.

The remainder of the book, concerning integration and with an early exposure to improper integrals, is acceptable, though exclusion of indeterminate forms is regrettable.

One thing is sure—this text is both unconventional and fascinating. It will probably cause many departments at least to reappraise their present offerings and produce some subsequent revisions.

This reviewer would hazard the guess that this book should be used primarily with selected students having solid high school training—preferably along the same general lines employed by this book. Surely, the typical freshman student would find the text anything but easy.

In any event, the authors have presented a real challenge to both student and instructor. May all concerned find its use a profitable experience!—*W. F. Brenizer, San Bernardino Valley College, San Bernardino, California.*

Plane Trigonometry, A. W. Goodman (New York: John Wiley and Sons, Inc., 1959). Cloth, xvii+267 pp., $4.50.

If this textbook is intended for use primarily in the secondary school it has more in its favor than if it is considered for use in a college course in trigonometry. The material it contains seldom goes beyond the elementary stage. It is disappointing to find a new textbook which fails to include more material integrating a particular course with its closely related fields of subject matter. Also, the chapter on complex numbers and De Moivre's Theorem is rather brief for modern demands.

There is just enough integrating material included in this textbook to tantalize the instructor with the possibilities inherent in its use. Examples of this are: (1) a paragraph dealing with "A Principle of Duality for Trigonometric Identities"; (2) the use of the distance formula for the derivation of the law of cosines; and (3) when mention is made of "In deriving a number of formulas for the calculus the ratio $(\sin \theta)/\theta \cdots$." The last idea could have been pursued further, and, either in this chapter or in the one dealing with definitions, the series that may be used to calculate values for the trigonometric functions could be presented.

The editors assert that "the discussions are simple yet detailed." True, but in the presentation of these discussions simplicity is achieved at the expense of content. For example, on pages 19–22, there is a discussion of the variations of the trigonometric functions that employs only four right triangles with no mention of angles greater than 90 degrees. It is not until pages 130–143 that variations of the values of the trigonometric functions as the angle varies from 0 degree to 360 degrees are considered.

An explanation for the organization of the material in the textbook is that it is "arranged in the historical and natural order, proceeding from the specific to the general." When the "specific" and the "general" are separated by more than forty pages of other subject matter, such as logarithms, students may have difficulty arriving at the proper generalizations. The reference here is to the definition of the trigonometric functions of acute angles, page 14, and the delay until page 59 for a consideration of the trigonometric functions of a general angle. There is a further lag of 100 pages before inverse trigonometric functions are considered. Such a disjointed presentation of closely related materials tends to prevent the students from obtaining a concept of the subject matter's interrelatedness.

A second instance of the separation of materials which could be logically combined is the chapter on oblique triangles in the first half of the book, and, in the latter portion of the book, the chapters that are concerned with the tangent of half an angle formulas, areas, and vectors. If all this material were in one unit, the students could have a comprehensive view of the general triangle, while the instructor could omit the parts he considered extraneous.

Some specific criticisms with regard to phraseology or use of terms are: (1) "cancellation and its inverse," rather than *division and its inverse;* (2) "a distance formula from analytic geometry" when *a distance formula* would suffice; (3) "transposing" rather than *solve the equation for . . .* ; and (4) if the case of a triangle with only three angles is to be considered, *do not determine a unique triangle* would be more meaningful to the average student than "cannot be solved."

Some recommendations for this textbook are: (1) the type is easily read, and the diagrams are attractive as well as functional; (2) the chapter "Some Preliminary Notions" contains ideas and symbols helpful for beginning students (the paragraphs on "Approximations and Significant Figures" might well be included in this chapter); (3) the introduction to the chapter is concerned with the addition formulas and the use of the unit circle in the derivation of these formulas.

It seems to this reviewer that this book would have better potentials in the hands of an experienced teacher than when used by an inexperienced teacher. The teacher with experience would recognize the parts that need supplementing as well as those that should receive little emphasis.—*Virginia Felder, Mississippi Southern College, Hattiesburg, Mississippi.*

Edited by Joseph N. Payne, University of Michigan, Ann Arbor, Michigan, and William C. Lowry, University of Virginia, Charlottesville, Virginia

Course requirements and grading

by William C. Lowry

There are some teachers who feel it is necessary early in a course to tell the pupils precisely what will be expected of them throughout the year and to specify for the pupils a definite, objective plan on which their grades will be based. Among the many other activities during the first day of school, then, a teacher of this persuasion also makes a rather detailed statement of the requirements in the course on which the pupils will be graded. For example, he may, with some discussion and elaboration, list the requirements as: a homework assignment each day, a quiz every Friday, an hour examination at the close of every grading period, and a notebook to be turned in at the end of each grading period. He then carefully describes how the grades and scores obtained on the homework, quizzes, tests, and notebooks will be combined for each grade report and for the final course grade.

There are advantages in starting a course this way. At least some pupils gain a feeling of security. They know precisely what they will be graded on. They know when and where they must make their greatest efforts. They feel comfortable in such an arrangement. There are certain advantages for the teacher, too. If he sticks strictly to his plan and his requirements, if his quizzes and examinations are objectively designed and graded, most pupils will feel he is being "absolutely fair." It is easier for him to determine grades, since the grades for the periodic reports will be arithmetically computed in a straightforward manner from the various scores he has collected on the pupils during the period.

The procedure needs closer examination, however. The advantages can, and often do, turn out to be disadvantages when viewed in the light of the objectives of a good course in mathematics. Consider the fact that within every class there is a range, sometimes quite great, in ability and achievement. An able pupil may listen to the teacher's list and sigh with relief. He is probably aware by the time he is in high school that he can easily keep near the top of the class if he knows precisely what is required in the way of work in the course. He may even be able to estimate roughly how much of his time "this course is going to take." There will be no need to put out any effort beyond that.

There is another type of able pupil, however, who will not make out so well. He rebels at set patterns. To him, grading on neat notebooks and homework—often busy work to be completed in detail—is falderal. An unyielding set of requirements to be met in order to obtain passing grades in the course acts as a strait jacket for him. The sameness of the week-by-week activities becomes quite dreary and monotonous and steals from him any enthusiasm that he might have for the work. Teachers are notoriously poor at picking out able pupils of this kind. In fact, such pupils are sometimes classified by teachers as slow learners and "trouble makers." In many cases grades reflect quality of performance

on a required set of activities which may not, in fact, be indicative of how capable the pupil is in mathematics and how much mathematics he is learning.

Insofar as getting good grades is concerned, the procedure is perhaps most advantageous for the average child who has learned to make the most of his abilities by sheer hard labor and by following directions closely. Given time, he can look almost as good in such a restricted program of grading as the able pupil. The slow pupil may give up at the start. After all, he may have had several years of competing on this basis prior to the course, and any plan of grading which treats him as a statistic will motivate him little, if at all. In short, as "fair" as this approach may seem, it is "unfair" because it makes little or no provision for individual differences in ability and achievement nor for differences in how children go about learning mathematics.

In terms of certain desirable objectives for the study of mathematics this procedure often fails miserably. Among such objectives are those which have to do with the outcomes of how pupils learn mathematics, outcomes which result from those experiences which lead to pupil discovery, to pupil development of mathematical ideas, and to freedom in attacking problem situations. Even though he may give verbal adherence to such objectives of instruction, the mathematics teacher who outlines a plan for grading like the one given earlier finds his quiz and examination questions gravitating toward items of recall and computation and his homework assignments becoming mainly practice sets. Such tests are easy to construct in an objective fashion, and such homework is easy to grade objectively. The marking of this work yields numerical scores which can easily be averaged in some way for purposes of the grades on the periodic reports. Although it does not happen invariably, the tendency is toward the tell-demonstrate-practice-test type of learning situation and away from activities leading to pupil initiative, development of ideas, and freedom in working with mathematics.

Another criticism of the approach described in the first paragraph is that it is not a positive approach. To give such a list of requirements the first day puts the emphasis on the grades to be received—not on the real purposes of the course—and, in effect, acts as a threat to the pupils. The course gets under way on a negative, foreboding note. The teacher does not need to belabor the points of requirements and grades. It should become evident to the pupils by the way the teacher goes about the work that the business at hand is learning mathematics. The venture is to be an exciting one and a pleasant one. It is to be a case of teacher and pupils considering mathematics together, not a case of teacher versus pupils in a situation dreaded by all. The teacher will judge how well the pupils are learning mathematics. The determination of grades will be based in part on test and quiz scores—every high school pupil knows that—but the teacher will make evaluations on other bases, also. All this is communicated to the pupils by the way in which the teacher goes about things, not by what he says the course and the requirements will be, but by what he does.

Some teachers are able to create a friendly, professional atmosphere in the classroom almost immediately. They rapidly, but thoroughly, complete the details necessary the first day of school in an easy, competent manner which demonstrates that the details are secondary in importance to the main purpose of the course. The study of mathematics is then started immediately. One teacher may start by proposing a good problem. This problem should require a lot of thought but little computation. It should be one that leads to a discussion in which the pupils have a chance to make known their thinking about the situation. It should illustrate important mathematical concepts and lead into the first topic to be taken up in

the course. By his questions and remarks, the teacher makes clear to the pupils that everything they do and say is being considered and evaluated in a friendly, helpful way.

Another teacher may accomplish a positive approach with a review lesson the first day. But it certainly is not accomplished by saying, "Take the problems on pages two and three for tomorrow. These are review problems. You know how to do them, so I expect all of you to complete them by tomorrow." It is best accomplished by putting the review material in an entirely new light. In a seventh-grade class the teacher may open the review of long division the first day with an analysis of the algorithm or by showing a different algorithm, but not by assigning twenty review examples in long division with the sole purpose of making certain there is some homework the first night of the new term. The teacher assures the pupils he is aware that they already know a lot about the review material, but they do not know all there is to know about it. They are more mature now than they were when they first studied the material; in the review they are going to work with ideas that are familiar to them but at a more mature level and in a new and unfamiliar way.

One may also criticize the demand for extreme objectivity in grading. Many teachers feel they must have evidence in terms of scores in a class record book to support the grades they give. They feel that a grade which includes the teacher's evaluation of how well a child can approach the learning of mathematics independently and how much power he has in making his own development of mathematical ideas is practically impossible to determine objectively, and, hence, should be avoided. But, even assuming such judgments are made somewhat subjectively, is not the teacher the best judge in the classroom of progress in these directions? He should not be afraid to make evaluations of the pupils on these "less objective" factors and he should not be afraid to let those evaluations be reflected in the grades. Every professional person must make judgments on the basis of his training and experience.

New teachers and student teachers often feel that they must state clearly what they expect of pupils, that the realization of these expectations must be capable of being determined in a strictly objective manner, and that once they have listed explicitly what is required in the course and how it will be graded, the work will proceed smoothly. But, as logical as this approach appears on the surface, these teachers may be thwarting their own objectives regarding learning and grading; they may actually be laying the groundwork for outcomes other than those desired.

"One inevitable result of this explosive development of mathematics has been the creation of new subject matter. Such fields as mathematical logic, probability and statistical inference, topology, and modern abstract algebra are largely or wholly the products of recent mathematical research."—"Program for college preparatory mathematics," *Report of the Commission on Mathematics, College Entrance Examination Board, New York, 1959, p. 1.*

• NOTES FROM THE WASHINGTON OFFICE

New developments in the NCTM Office

by M. H. Ahrendt, Executive Secretary, NCTM, Washington, D. C.

As members of the NCTM would expect, the continuing and apparently increasing interest in the problems of mathematics education requires continuing adjustments and adaptations in the NCTM Office. We have made a number of changes recently which may be of interest to you.

New Staff Members. We have been fortunate in being able to add two professional workers to the office staff. J. James Brown, a former high school teacher in Columbus, Ohio, has joined the staff as professional assistant. Mr. Brown has been an active member of the NCTM throughout his teaching career. He has been especially interested in local association activities. He will assist the executive secretary with both the business and professional management of the NCTM Office. With his assistance we hope to be able to do our present work better and more comfortably and to extend the service of the office to other areas.

Miss Miriam Goldman has been employed to serve as editorial assistant. Miss Goldman has had extensive editorial experience, especially with technical and mathematical literature. Her first big project since joining the staff has been the preparation for the press of the manuscript for the 25th yearbook, *Instruction in Arithmetic.* With her editorial skills we expect to maintain, and if possible to raise, the high standard of editorial and printing quality that we have sought for the NCTM publications.

We have also had to increase the size of the secretarial and clerical staff. We feel that we have been fortunate in finding employees of high quality to process your orders for memberships and publications and to take care of the myriad of business details involved in serving you efficiently.

New Office Space. It has been a habit of the NCTM Office since we affiliated with the NEA to outgrow our office space every year or two. This fall for the fifth time since 1950 we expanded into larger quarters. Unfortunately there was no single area available in the new NEA Building large enough to contain our entire enlarged office staff of 16 persons. Therefore we were forced to divide the office into two parts, placing our Billing and Order Section on another floor. We have devised means of communication between our two office areas so that this division of the office should not result in any loss of efficiency.

New Equipment. The NCTM Office has operated during past years with very little specialized office equipment. Typewriters and standard office furniture are lent to us by the NEA. Specialized items such as duplicating machines must be purchased by us. The Board of Directors, realizing that we had reached the point where additional equipment would contribute greatly to the ease and efficiency with which we did our work, placed a much appreciated allowance for equipment in the budget for this fiscal year. This allowance has enabled us to obtain the following items: an automatic postal mailing and metering machine, a computing postal scale, a photocopy machine, three electric typewriters,

an additional dictating machine (for the professional assistant), and an office safe.

New Membership Growth. While we are mentioning new developments, it seems appropriate to add another which you, the members, have given us. Normally the membership level remains practically constant during the summer months, and the first evidence of membership growth during the new school year appears in the fall mail. This year, however, our preliminary totals indicate a membership growth of 10 or 15 per cent during the summer months alone. If the rate of growth continues to accelerate, as we believe it will, we shall be even more glad that we were able to obtain this new staff, space, and specialized office equipment.

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N. W., Washington 6, D. C.

NCTM convention dates

JOINT MEETING WITH MATHEMATICAL ASSOCIATION OF AMERICA

January 30, 1960
Conrad Hilton Hotel, Chicago, Illinois
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D. C.

THIRTY-EIGHTH ANNUAL MEETING

April 20–23, 1960
Statler-Hilton Hotel, Buffalo, New York
Louis F. Scholl, Board of Education, Buffalo 2, New York

JOINT MEETING WITH NEA

June 29, 1960
Los Angeles, California
M. H. Ahrendt, 1201 Sixteenth Street, N. W., Washington 6, D. C.

TWENTIETH SUMMER MEETING

August 21–24, 1960
University of Utah, Salt Lake City, Utah
Eva A. Crangee, Board of Education, Salt Lake City 11, Utah

Other professional dates

Women's Mathematics Club of Chicago and Vicinity

January 30, 1960
Stouffer's Restaurant, Randolph and Michigan, Chicago, Illinois
Sarane Starr, University of Chicago High School, Blaine Hall, Chicago 37, Illinois

Mathematics Section, New York Society for the Experimental Study of Education

February 13, 1960
1056 Thompson Hall, Teachers College, Columbia University, New York, New York
John A. Schumaker, Secretary, Montclair State College, Upper Montclair, New Jersey

NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Report of the Nominating Committee

At the meeting of the Board of Directors of the National Council of Teachers of Mathematics held in Ann Arbor, Michigan, the report of the Nominating and Elections Committee was approved. The nominees for the election to be held in the spring of 1960 are as follows:

President

Phillip Jones, University of Michigan
Henry Van Engen, University of Wisconsin

Vice President, Secondary Level

William H. Glenn, Pasadena, California
Eunice Lewis, University of Oklahoma

Vice President, Elementary Level

Joyce Benbrook, University of Houston, Texas
Clarence Ethel Hardgrove, Northern Illinois University

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A. Houston Banks, George Peabody College for Teachers
Irving H. Brune, Iowa State Teachers College
Eva A. Crangle, Salt Lake City Public Schools
Z. L. Loflin, Southwestern Louisiana Institute
Eugene D. Nichols, Florida State University
Robert E. K. Rourke, Kent School, Kent, Connecticut

Respectfully submitted,
Ida May Puett, Chairman

Joint Meeting of the National Council of Teachers of Mathematics with the Mathematical Association of America

Eighth Street Theater
Chicago, Illinois
Saturday, January 30, 1960

This meeting will be devoted to presentations, panel and audience discussions of several related problems of critical interest today to *both* high school and college teachers.

These problems include the Advanced Placement Program and such proposed revisions in the regular college preparatory program as the introduction to analytic geometry, the deletion of solid geometry, and the teaching of calculus in high school classes.

Speakers and panel members will include college, high school, and preparatory school teachers.

The program committee for the NCTM includes E. H. C. Hildebrandt, Harold Larson, Henry Van Engen, and Phillip Jones, chairman.

The committee on local arrangements is chaired by Irwin K. Feinstein and includes Louise Fisher, Ramona Goldblatt, Maude Bryan, and Florence Miller.

Room reservations may be sent directly to the Conrad Hilton Hotel which adjoins the theater and will be the headquarters for the Mathematical Association of America which is holding joint meetings with the American Mathematical Society earlier in the week.

How the slide rule fits into today's accelerated science and math program

ROBERT JONES, *Manager, Educational Sales Division, Frederick Post Company*

Increased government emphasis on secondary education poses this new challenge for mathematics teachers. Even before Sputnik I, our legislators in Washington took a long, hard look at the nation's secondary school curricula. The first result of their study: the National Defense Education Act of 1958. This Act in part calls for intensive emphasis on science and mathematics at the secondary school level. Much of the work which was formerly left for college must now be handled in high school. To meet these new requirements it's essential that pre-engineering or science majors learn the use of the slide rule *before* they get to college.

Unless they have guidance from you about what to look for in a slide rule, it's natural that your students would be attracted to the "bargain" slide rules that can be found almost anywhere selling for \$.75 to \$1.25. These rules have the basic scales, they *look* efficient to the unpracticed eye.

But are they *really* bargains? In the interest of better teaching and for the sake of your students, take a few minutes to evaluate these "bargain" slide rules for yourself. You'll undoubtedly find they have these weaknesses:

The graduations on "bargain" rules are printed or molded. Markings of this kind are often inaccurate, and almost always *temporary*. We've seen scores of rules on which markings fade after months of limited use. Is this a wise investment—at *any* price?

Another weakness of "bargain" rules is the basic material used. It will swell and contract under atmospheric changes. Once warped by these changes, the readings on these rules are often not dependable. The cursor is affected, too—it may bind or fit loosely and cause further inaccuracies.

At only slightly higher cost, we're sure you'll agree that the Post 10" #1447 Student Slide Rule (Mannheim type) is far more worthwhile for the average student. It sells for \$2.81 (classroom price) and offers sound value for every penny over and above the cost of "bargain" rules.

This slide rule is constructed of seasoned, laminated bamboo. Post has adopted bamboo because it is not affected by climatic conditions—it will not warp or shrink. The slide will not bind, stick or require artificial lubricants at any time.

The bamboo is laminated for extra strength as a further precaution against warping. Distortion is no problem with a bamboo rule of this quality.

Another feature of Post's Student Slide Rule is the engine-dividing of the graduations. Each graduation is precisely cut into the white celluloid face, making it a *permanent part of the rule.* The accuracy of the cuts is assured by modern machine controls.

The Post 1447 Student Slide Rule has the A, B, CI, D, and K scales on the face and the S, L, and T scales on the reverse side of the slide.

The cursor is framed in metal for durability, and a tension spring maintains the vertical position of the hairline. The hairline itself is etched in clear glass. The rule is furnished with a sturdy slip cap case and an easy-to-understand instruction booklet. An imitation leather case is available at slight additional cost. This rule serves the student dependably and accurately throughout his school years and on through his adult life.

Educators can help their students appreciably by advocating better slide rules (not necessarily expensive) for basic calculations. To prove our point, we'll be happy to supply you on a 90-day trial basis with one of these Post 10" Student Slide Rules, along with a complimentary catalog. Please address your inquiry to *Educational Division, Frederick Post Company, 3650 N. Avondale Avenue, Chicago 18, Illinois.*

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—integrates basic content from analytic geometry, beyond the limited stage customary in college algebras.

—develops trigonometry in the collegiate fashion, not wholly in consecutive chapters, employing results from analytic geometry with resulting simplification and improved logic.

—emphasizes logical procedures maturely, but simply, throughout.

—provides an elementary introduction to sets.

—gives the distinctly modern development of probability from the postulational standpoint for a finite sample space, with an introduction to random variables.

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—presents the theory in a sound fashion, but attains the desired objectives by intuitionally logical devices whenever analytical rigor would place the discussion above the level of the typical student.

648p. text

For each of the above texts, answers are supplied in the book for odd-numbered problems in exercises. Answers for even-numbered problems are available free in a separate pamphlet.

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